جـامعـة الشـارقـة
UNIVERSITY OF SHARJAH

Mathematical Modeling of Patterns in Islamic Arts

Mostafa ZAHRI

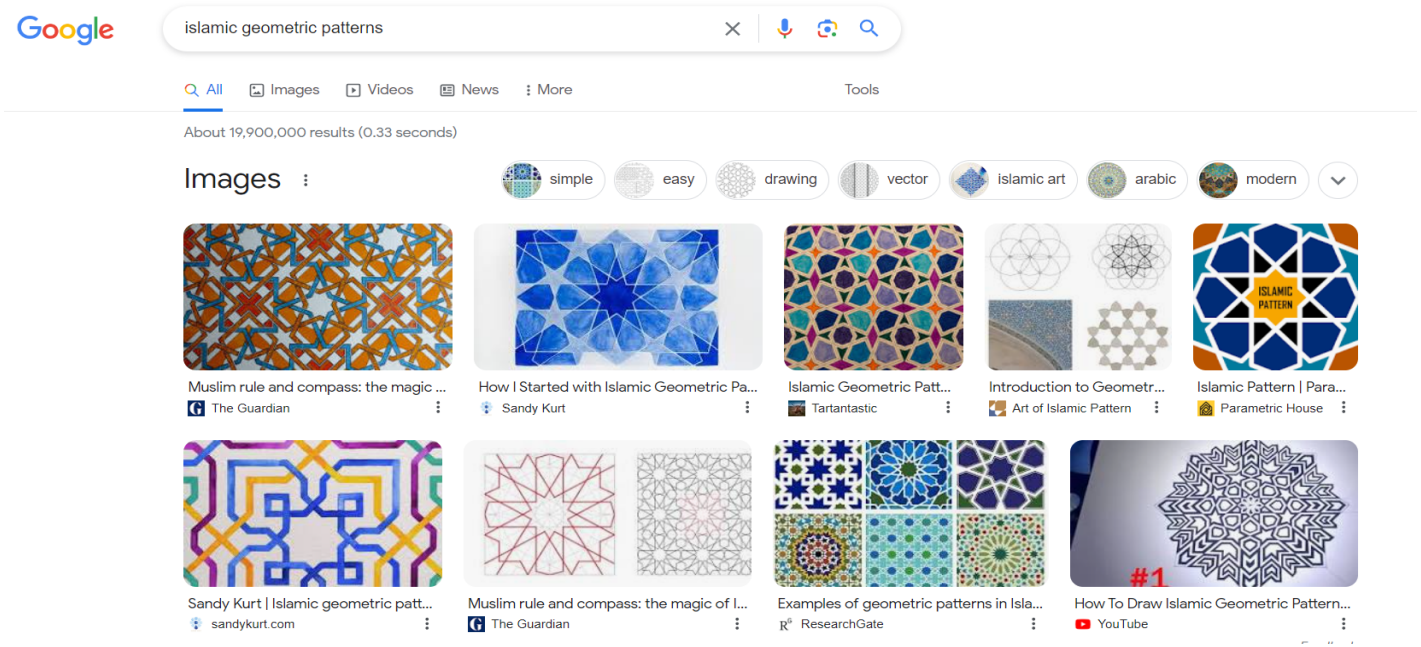
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University of Sharjah, United Arab Emirates

MATH-ARTS-DAY
UM6P Ben Guerir, Morocco
Dec. 19, 2023

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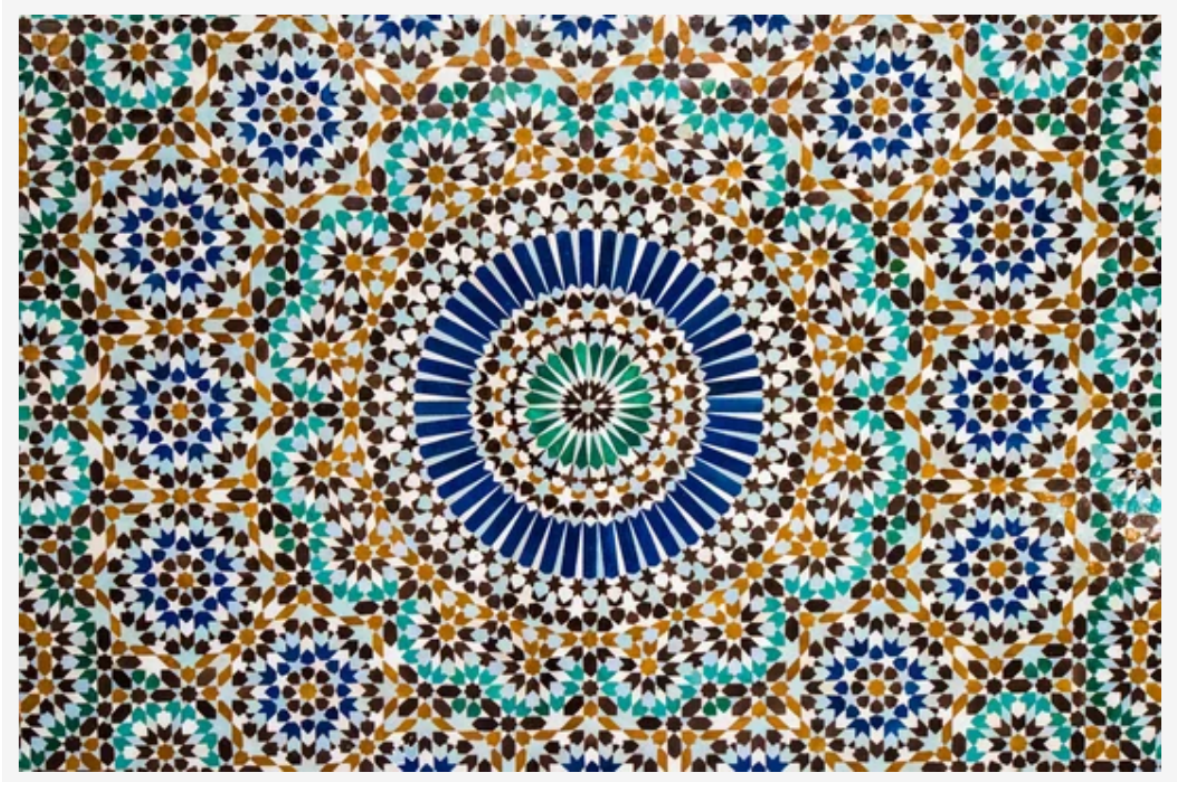
1. Moroccan Islamic Design
2. Periodic sequences of the first kind on the unit circle
3. Periodic sequences of second kind
4. Periodic sequences of third kind
5. Compiling 2D Patterns: Planar and Circular Tessellations
6. Space filling techniques
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8. Simulation of Rose Equations using Periodic Sequences

Islamic Design in the Web. (search words: islamic design) Related words: *Islamic complex geometry, - geometric pattern, - Design House, - Arts ...*

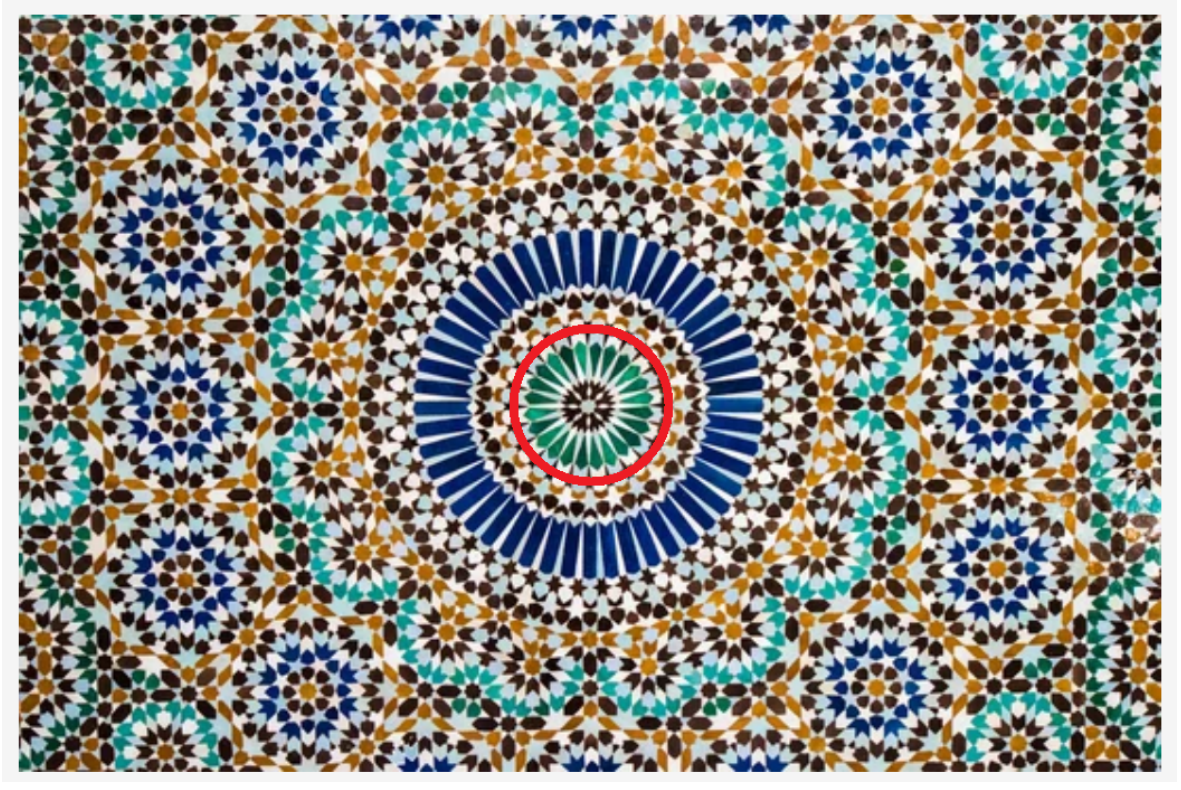


Zellige is mosaic tile work made from individually chiseled geometric tiles set into a plaster base. This form of Islamic art is one of the main characteristics of Moroccan architecture. It consists of geometrically patterned mosaics, used to ornament walls, ceilings, fountains, floors, pools and tables.
(source: *https://en.wikipedia.org/wiki/Zellige*)

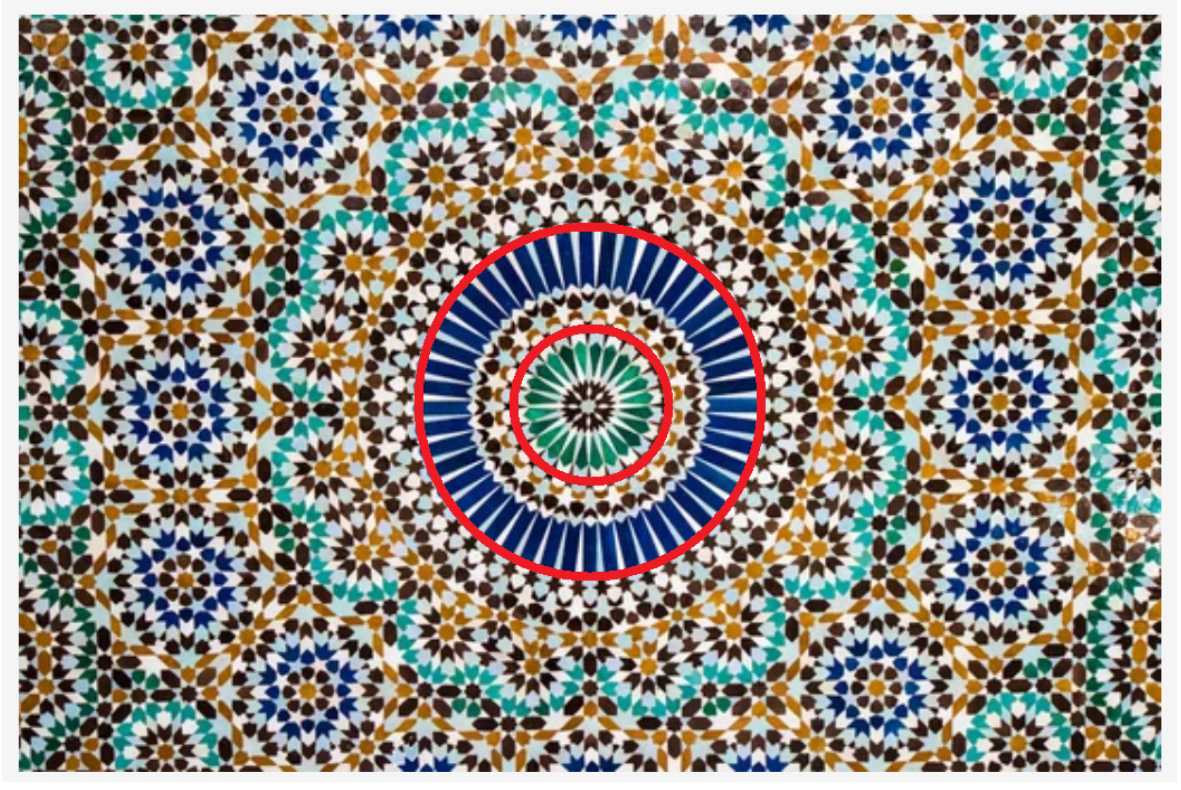
All sub-tiles fit into a circle, either completely or piece-wise:



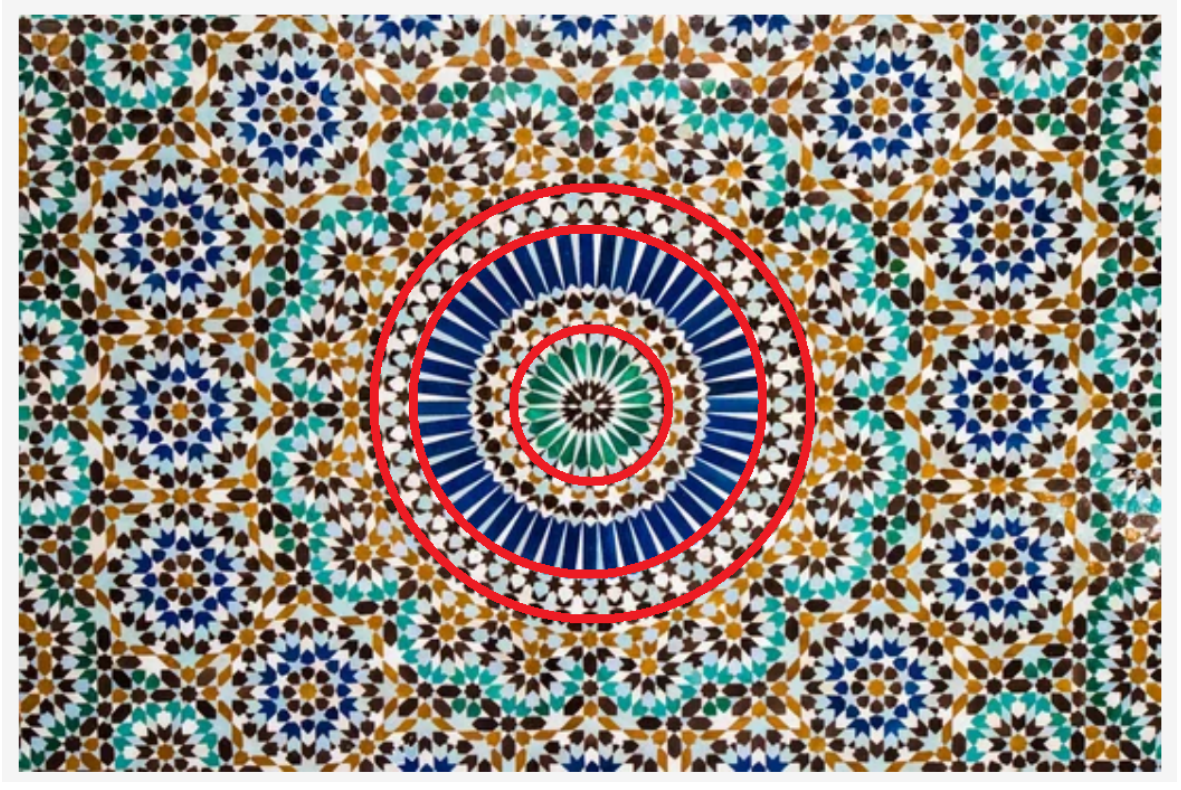
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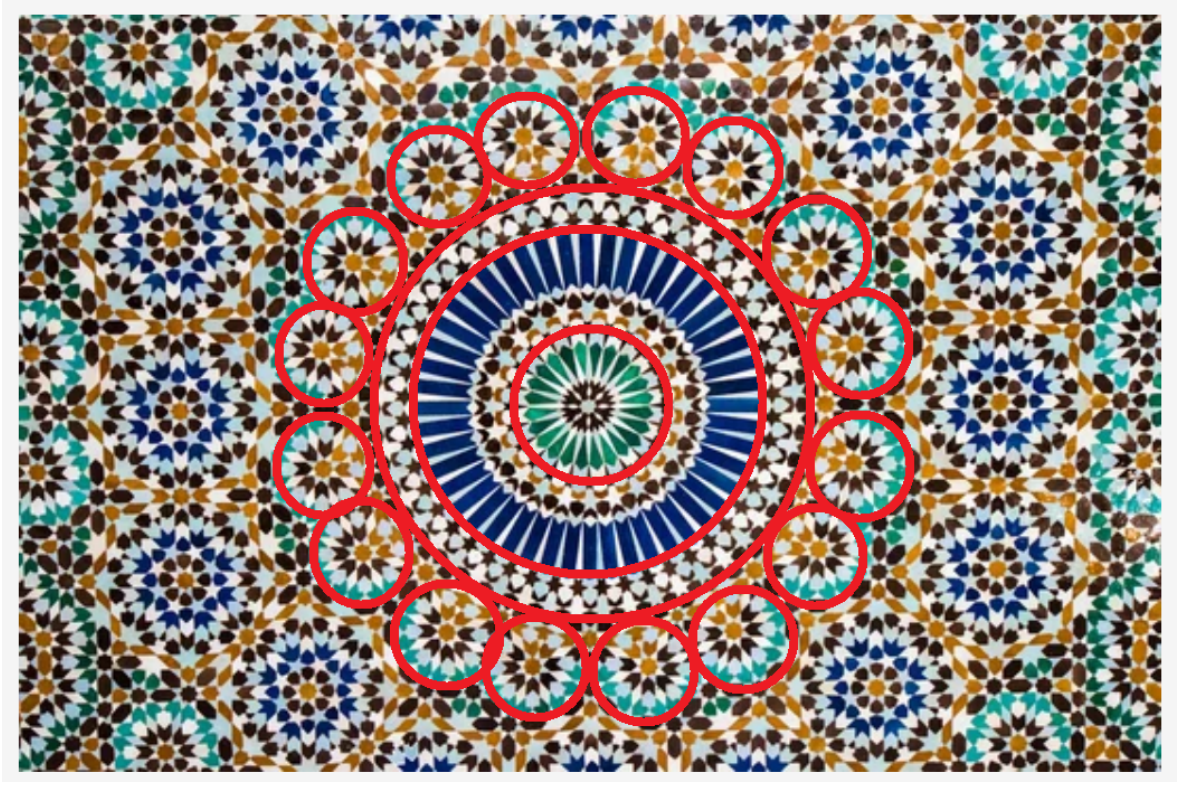
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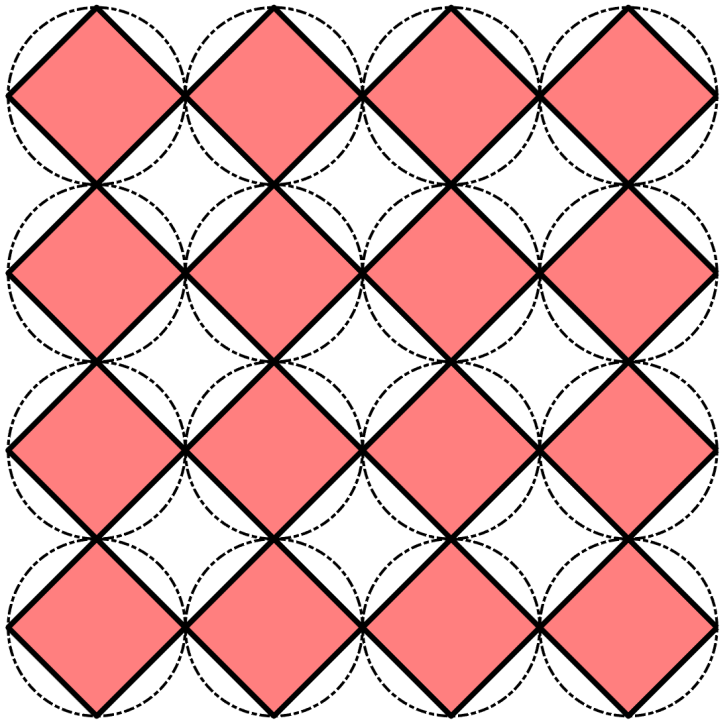
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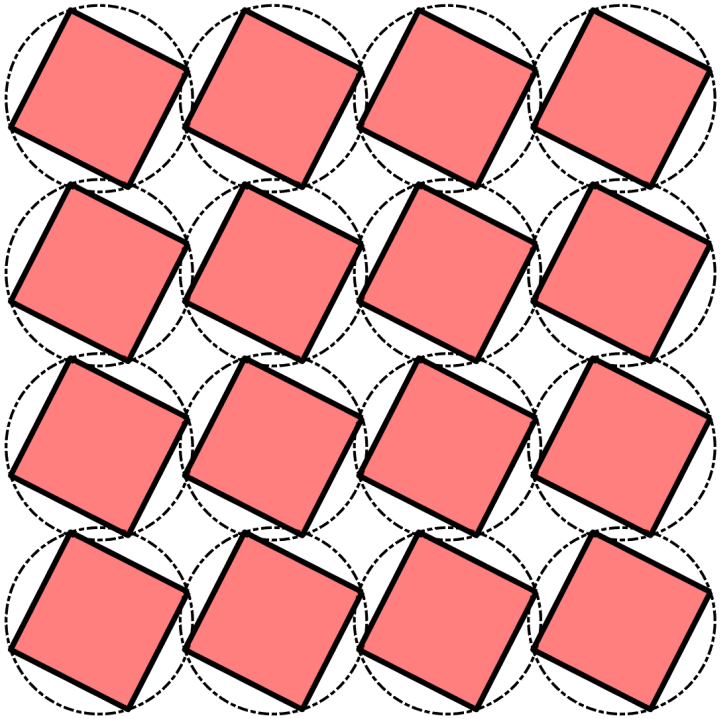
Could we find a mathematical model generating such pattern?



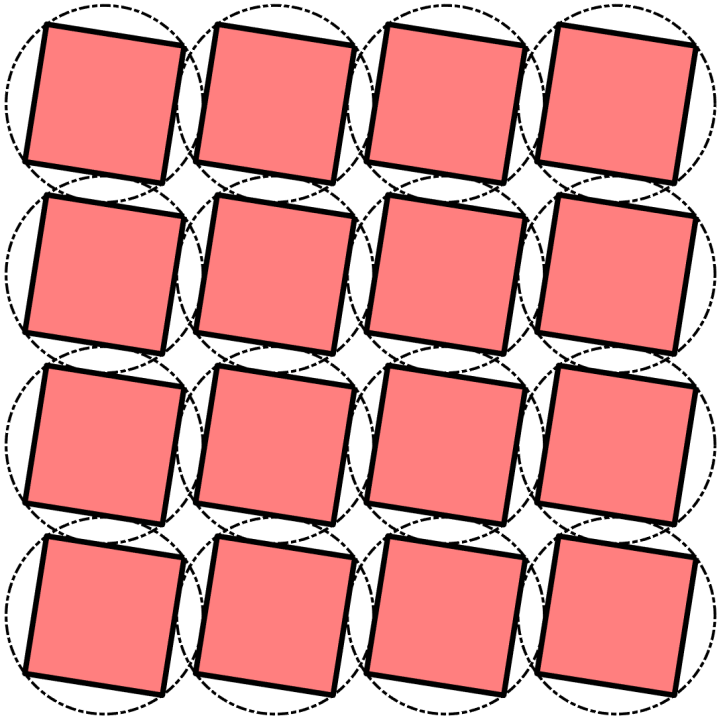
First remarks: Rotational, Horizontal and vertical displacement of Squares



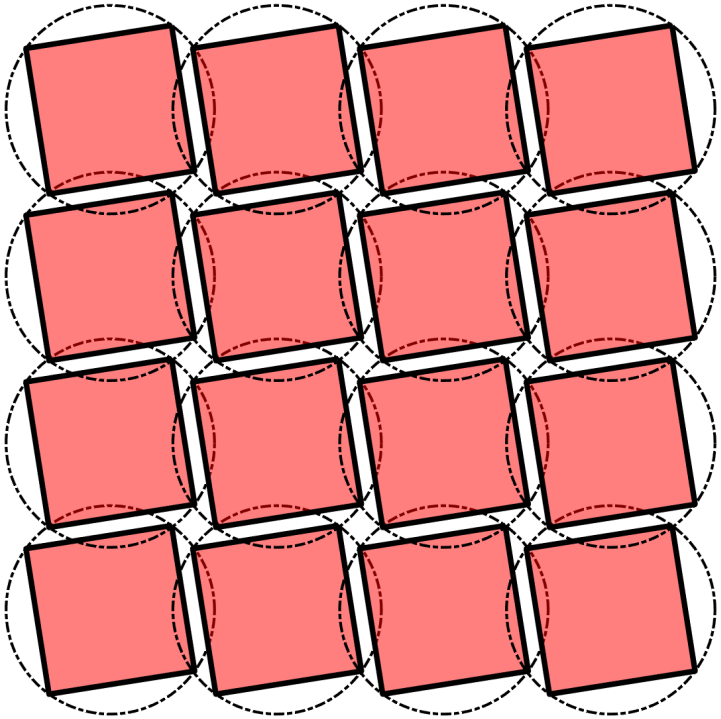
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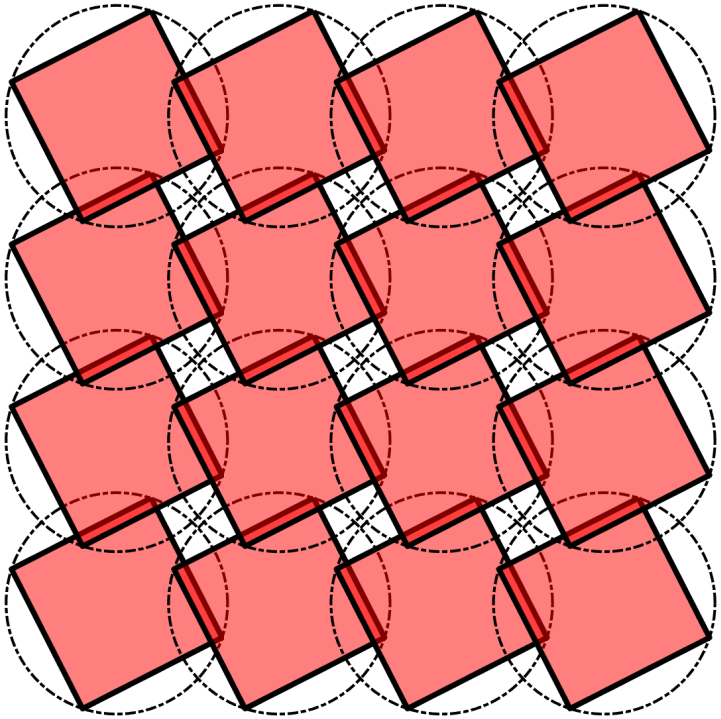
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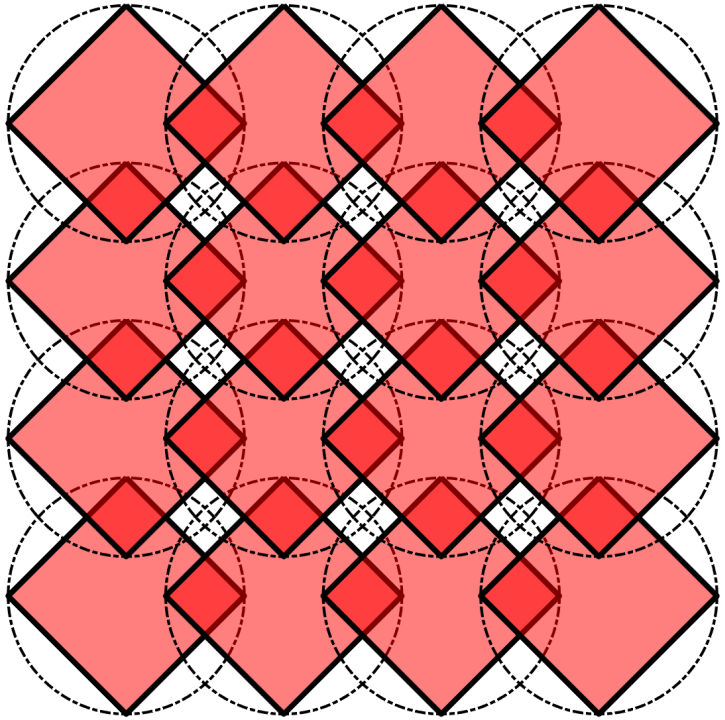
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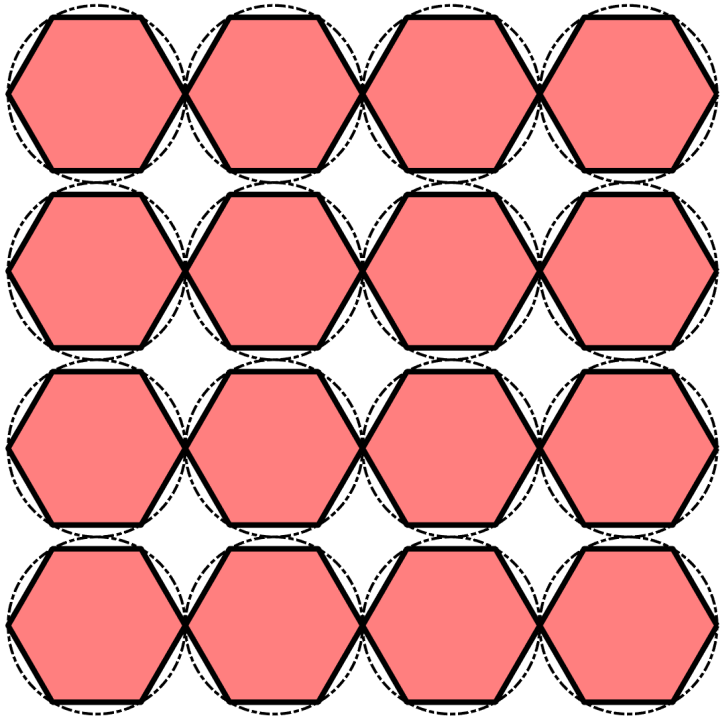
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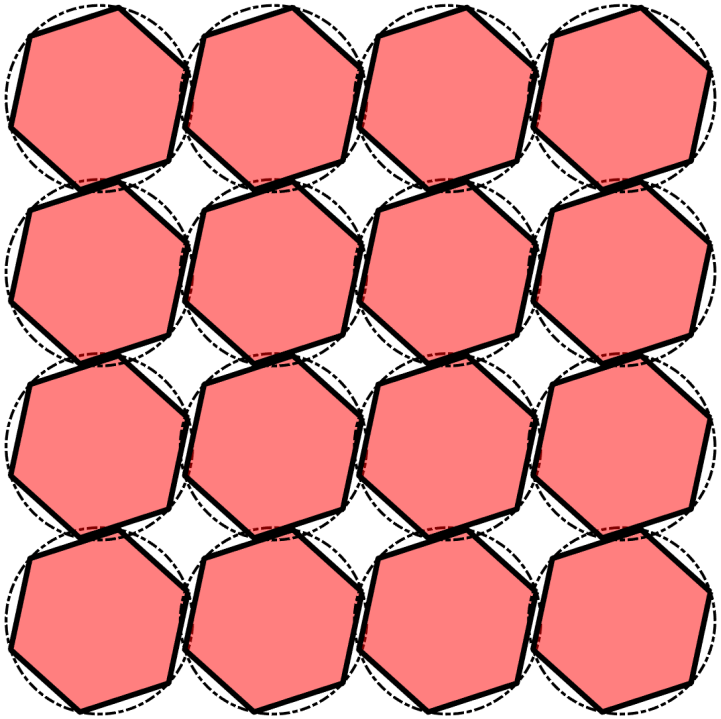
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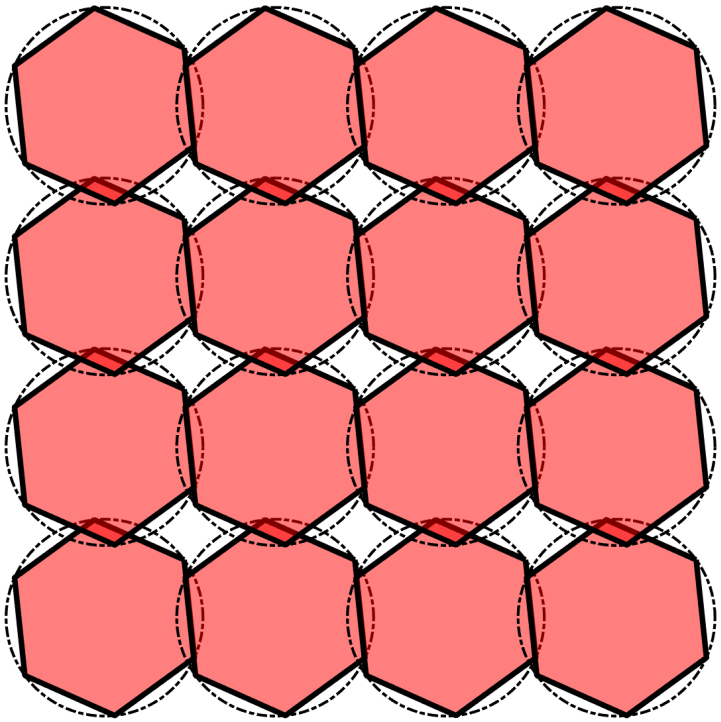
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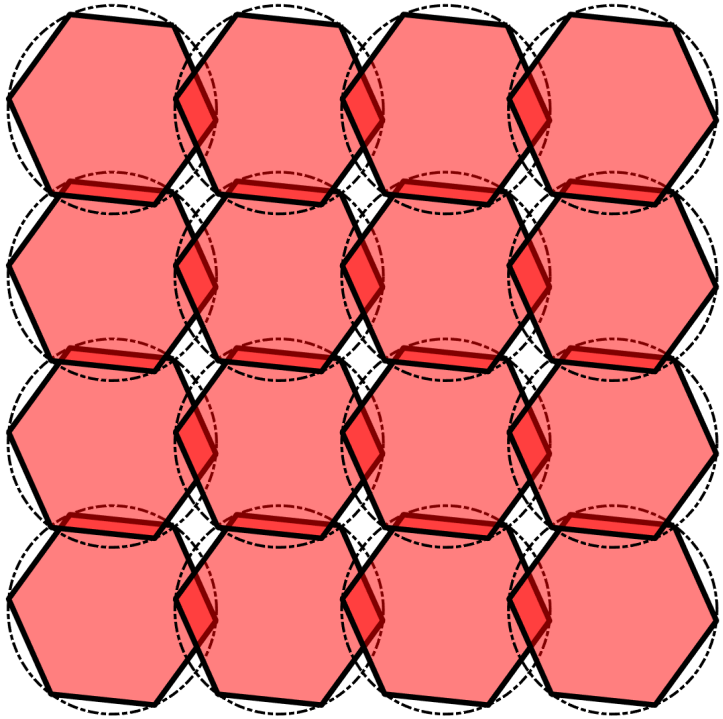
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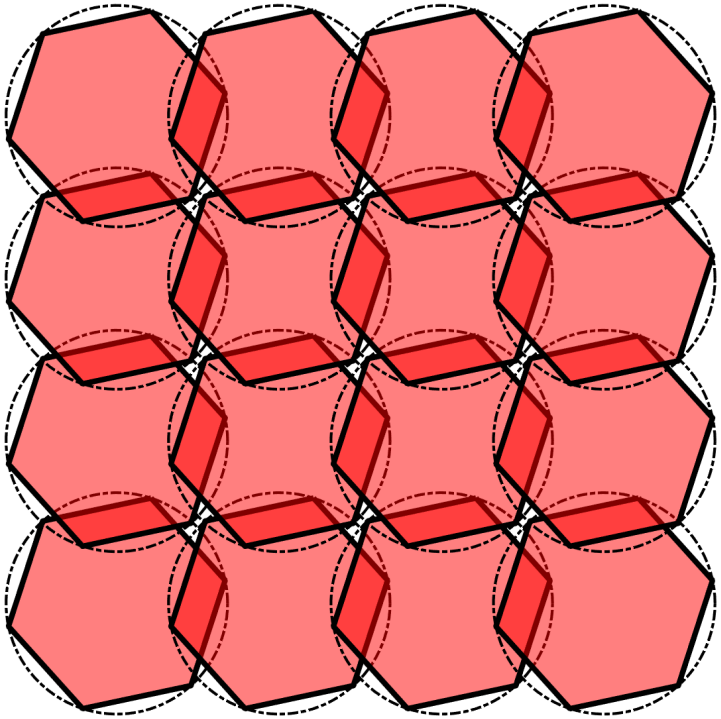
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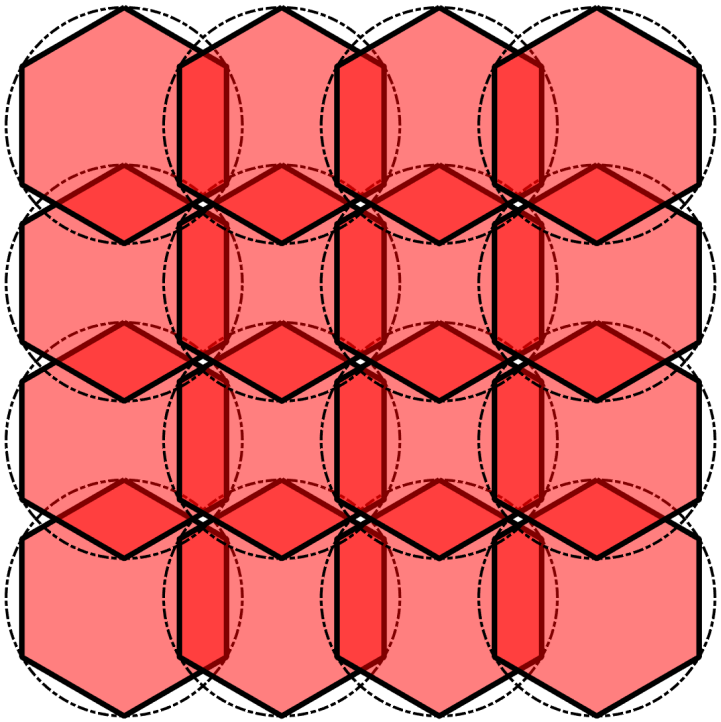
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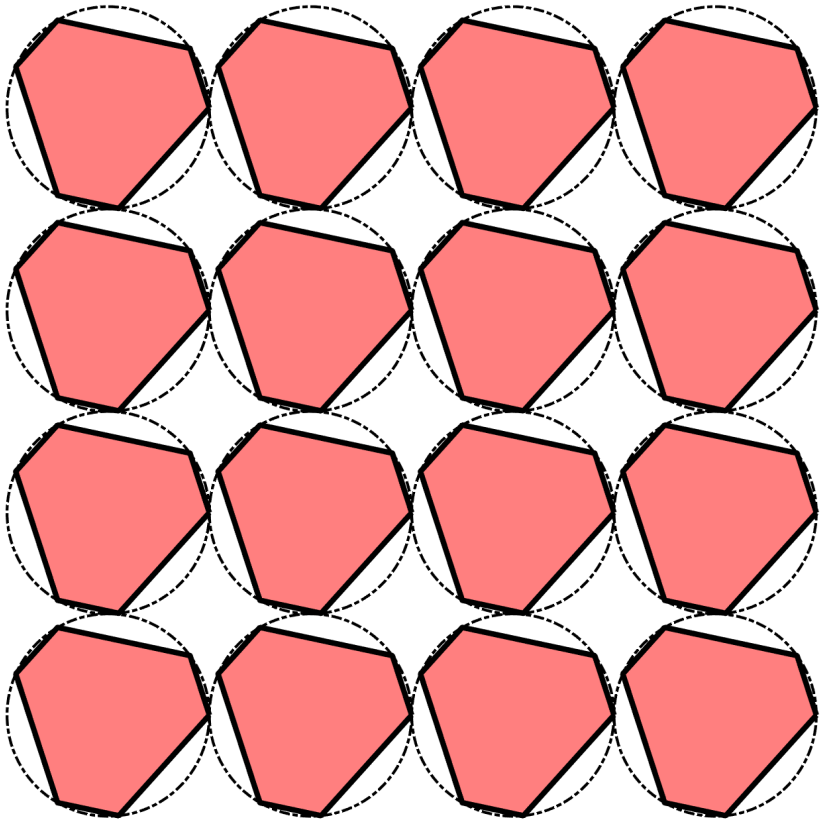
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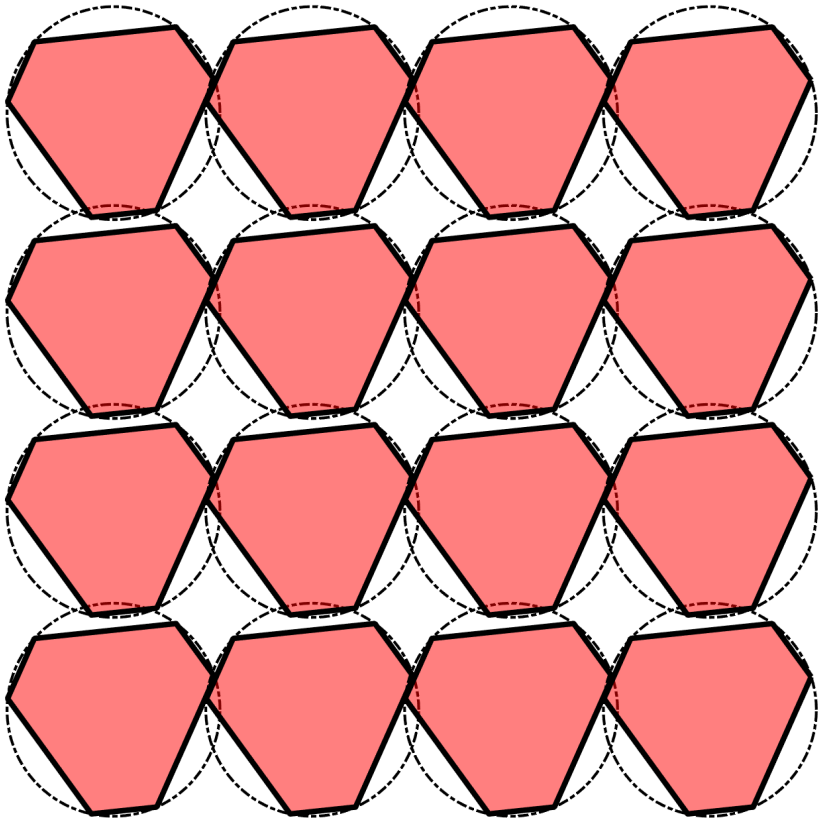
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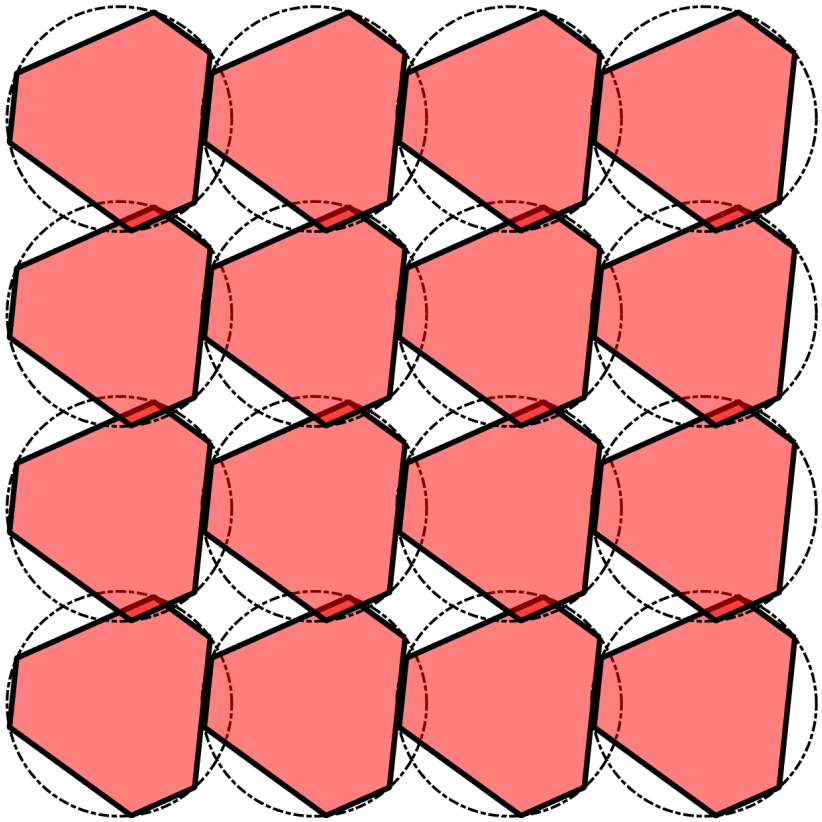
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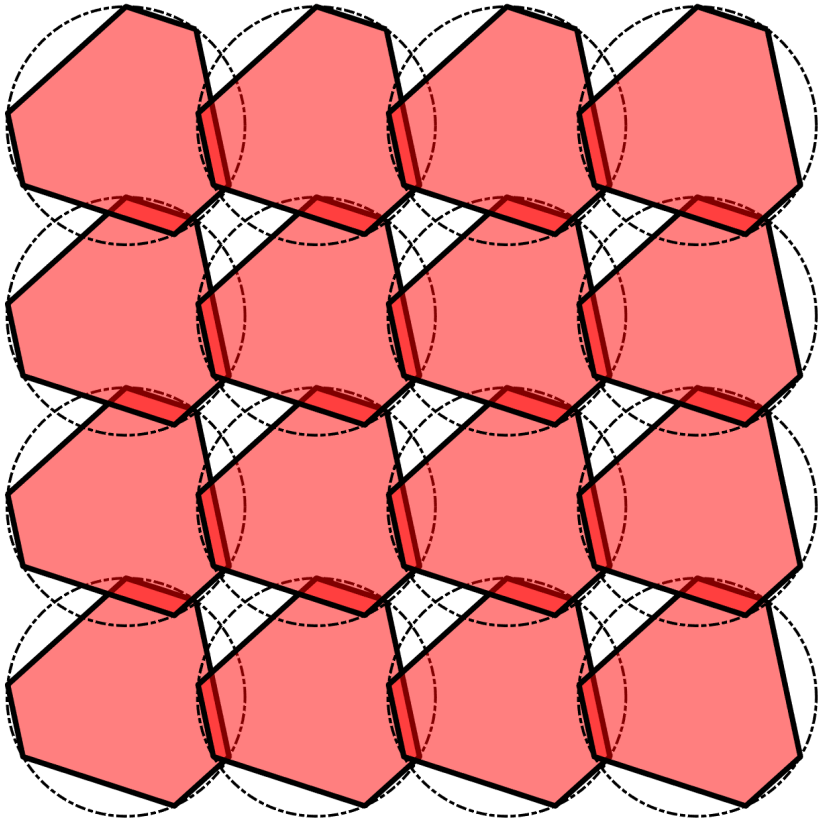
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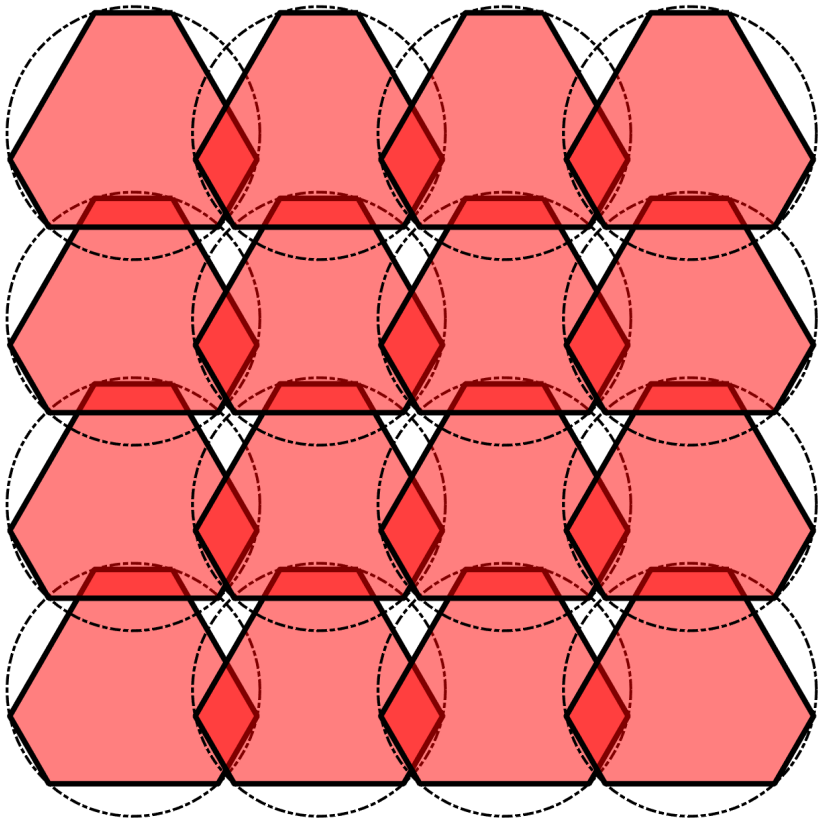
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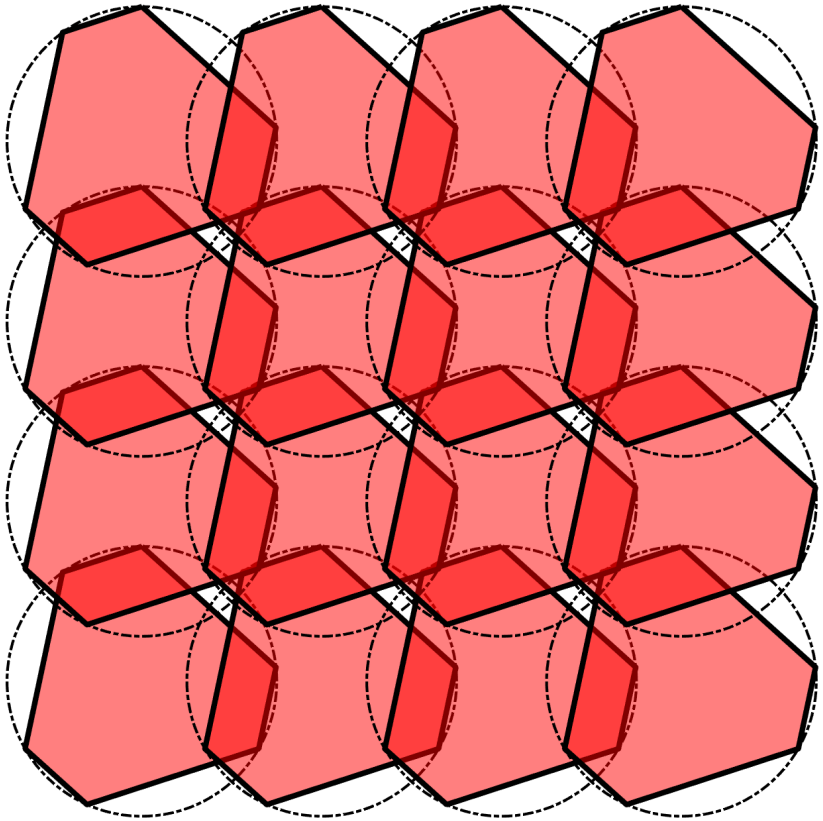
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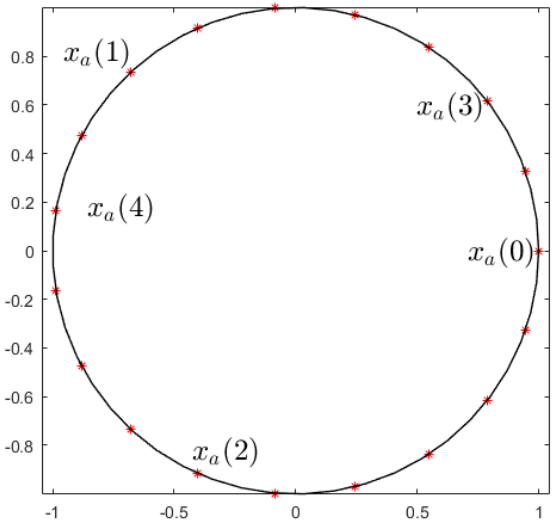


For $x_a(0) = \theta$, consider the real iterative sequence $x_a : \mathbb{N} \longrightarrow S^1 \mod 2\pi$ by:

$$x_a(0) = \alpha$$

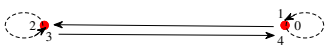
$$x_a(n) := \begin{cases} x_a(n-1) + a\pi & \text{if } n \text{ odd}, \\ x_a(n-1) - (1-a)\pi & \text{if } n \text{ even}. \end{cases}$$

- $x_a(n)$ are angles on S^1,
- $x_a(n)$ are vertices of a graph on $\mathbb{R}^2$,
- If k is the number of loops, such that $x_a(n) \neq x_a(0)$, then $x_a(n) \geq 2k\pi$,
- We are interested in the minimal index $n > 0$, such that $x_a(n) = x_a(0) = \theta$.



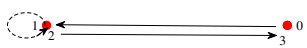
GEOMETRICAL ILLUSTRATION OF x_a

Relationship between a and the minimal index such that $x_a(n) = x_a(0)$ for $n > 0$:



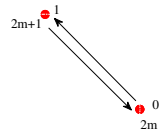
(A)

$a = 0;\ k = -1;\ n = 4$



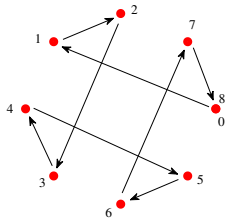
(B)

$a = 1;\ k = 1;\ n = 3$



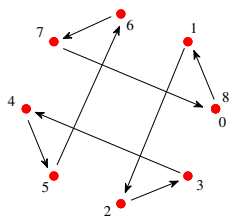
(C)

$a = 0.5;\ k = 0;\ n = 2$



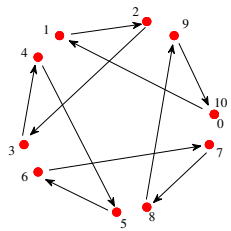
(D)

$a = 0.75;\ k = 1;\ n = 8$



(E)

$a = 0.25;\ k = -1;\ n = 8$



(F)

$a = 0.7;\ k = 1;\ n = 10$

- Case (a), (b) and (c) showing no patterns,
- Case (c), (d) and (e) showing geometric patterns,
- For which k number of loops, n number of vertices, we obtain a pattern?

CHOICE OF THE PARAMETER $a(k,n)$: STOPPING CONDITION

- We characterise the parameter a as a function of the number of loops k and the first-come back n_f closing the graph.
- Moreover, we distinguish between the odd and the even index as a come-back events:

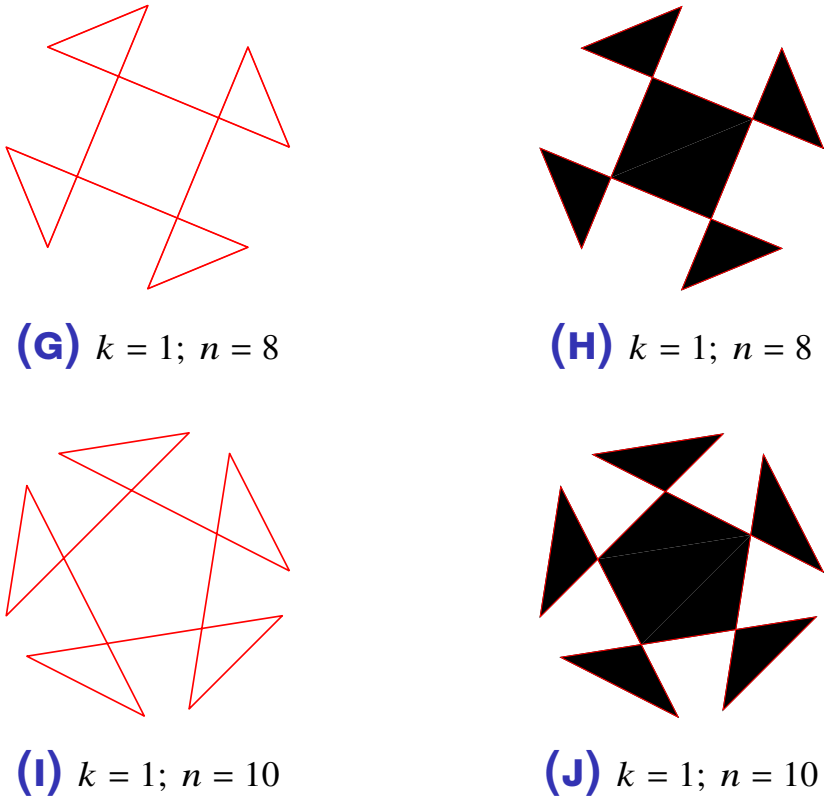
For construction of the parameter a, we define the come-back mappings:

$$x_a(0) = \theta$$

$$a(k,n) := \begin{cases} \frac{1}{2}\left(\frac{4k-1}{n} + 1\right) & \text{if} \quad n \quad \text{odd}, \\ \frac{1}{2}\left(\frac{4k}{n} + 1\right) & \text{if} \quad n \quad \text{even}. \end{cases}$$

EXAMPLES OF CELLS OF THE FIRST KIND

- Observing the trajectory (in $\mathbb{R}^2$) of the sequence x_a,
- From closed graphs to code is coloring colored cells:



(G) $k = 1;\ n = 8$ **(H)** $k = 1;\ n = 8$

(I) $k = 1;\ n = 10$ **(J)** $k = 1;\ n = 10$

FIGURE: The path of the periodic sequence $x_a(n)$ on S^1.

EXAMPLES OF CELLS OF THE FIRST KIND

The Cell $C(q, r, a, \theta)$ is a function of at least four parameters q centre of the shape, r radius, $a(k, n)$ governing parameter of the sequence, θ the initial guess:

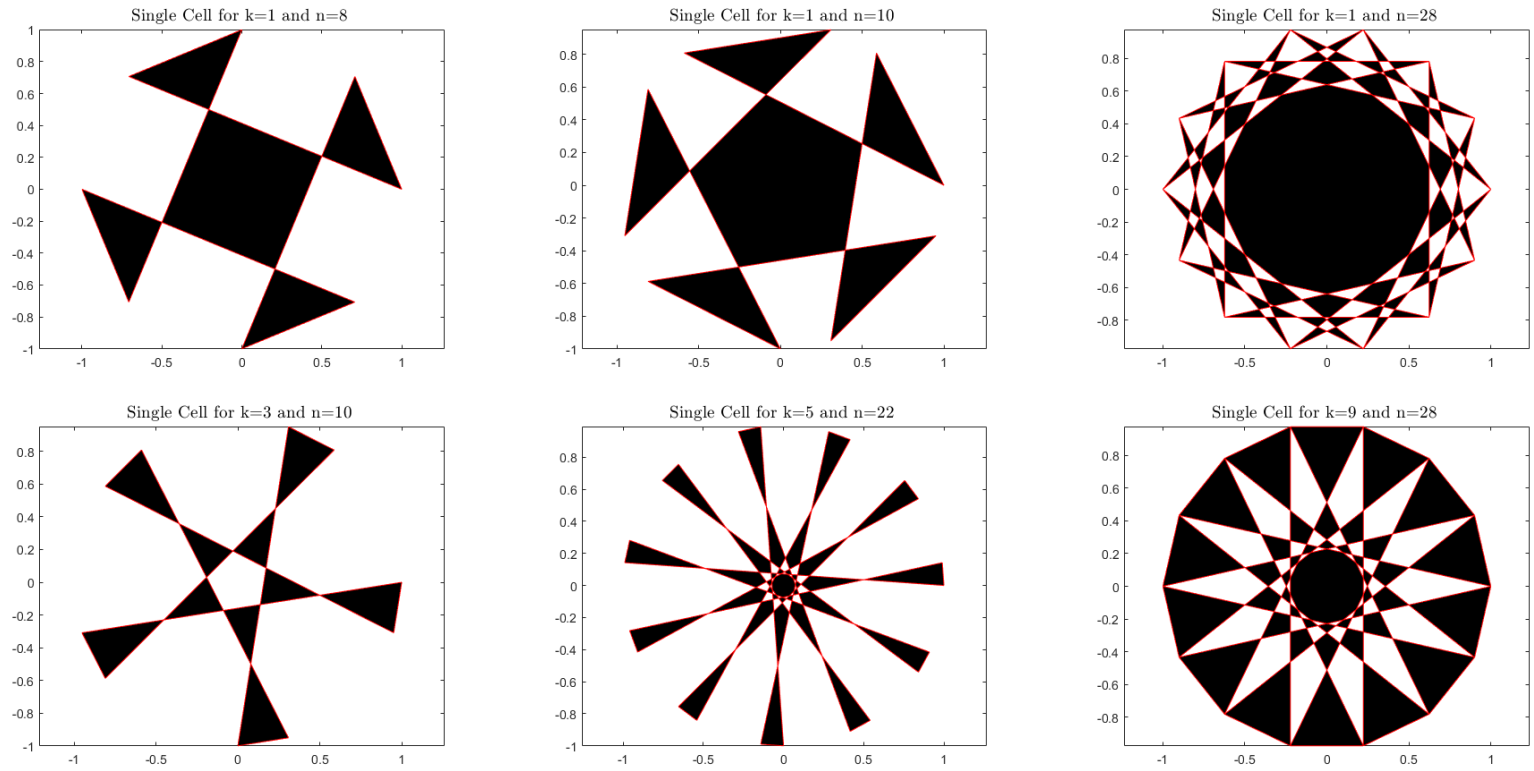


FIGURE: Cells for even n.

For n even the cells show symmetrical shapes.

EXAMPLES OF CELLS OF THE FIRST KIND

Odd numbers n

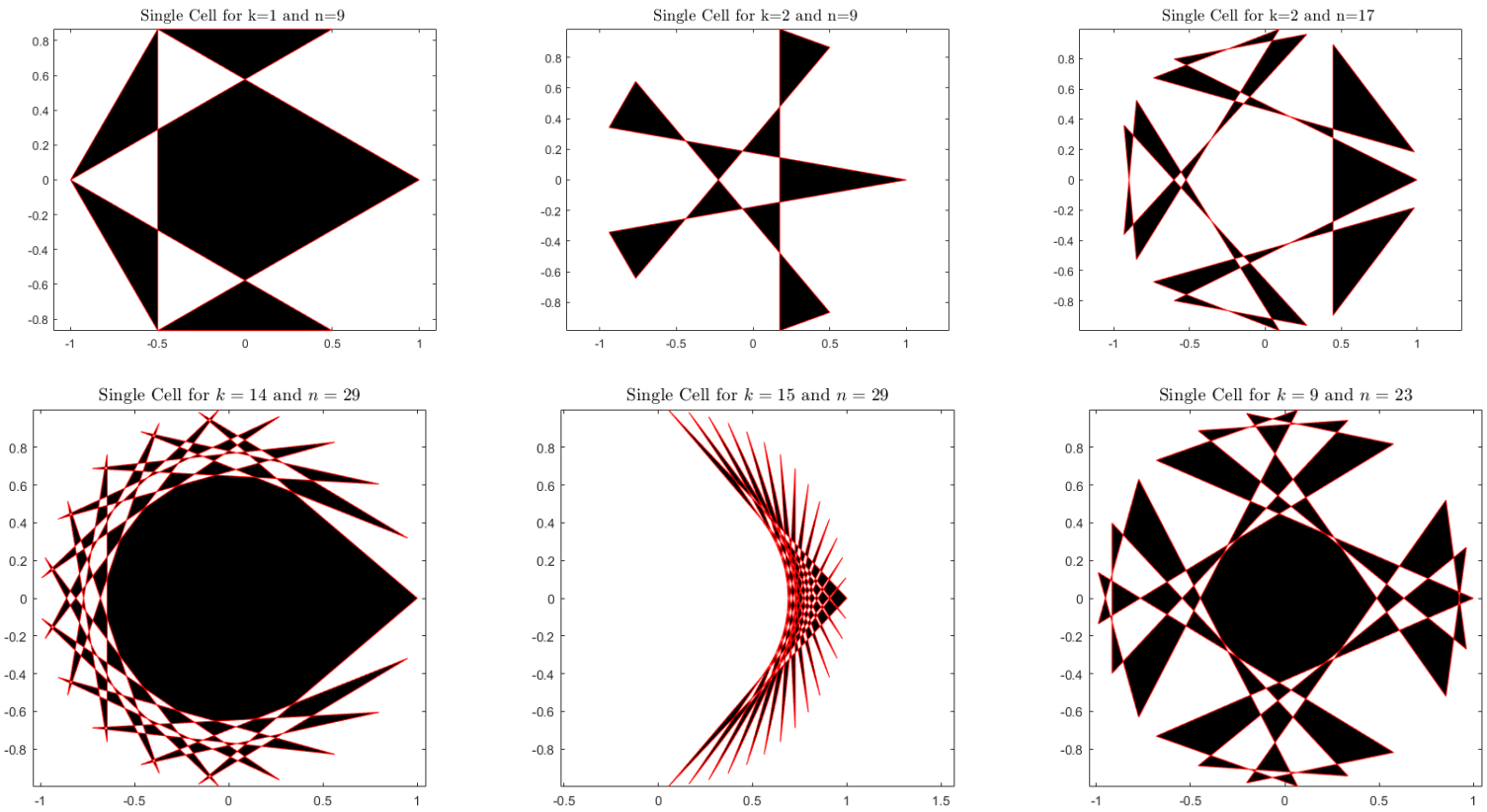
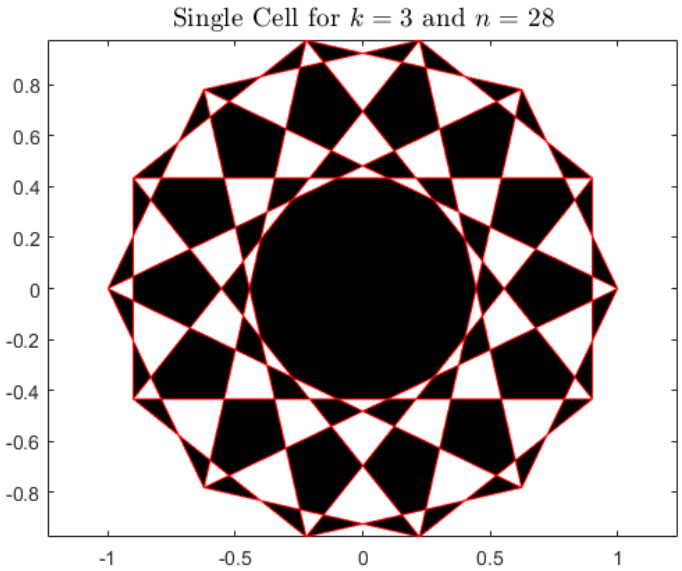


FIGURE: Cells for odd n.

For n odd the cells show in generally pseudo-symmetrical shapes.

Interesting mathematical questions ?



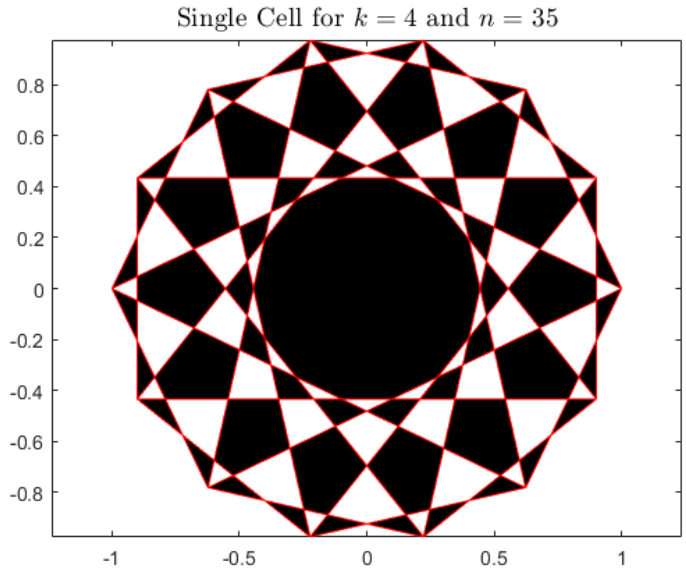


FIGURE: Example of equivalent Cells for even $(3, 28)$ and odd $(4, 35)$.

With respect to the sequence first kind:

- Why for $(3, 28)$ and $(4, 35)$ we obtain same shapes ?
- All vertices have same multiplicity. Why ?

For any initial value $x^{\alpha,\beta}(0) = \theta$, we iteratively define the sequence

$$x_a^{\alpha,\beta} : \mathbb{N} \longrightarrow S^1 \mod 2\pi$$

$$x_a^{\alpha,\beta}(0) = \theta$$

$$x_a^{\alpha,\beta}(n) := \begin{cases} x_a^{\alpha,\beta}(n-1) + a\alpha\pi & \text{if } n \text{ odd}, \\ x_a^{\alpha,\beta}(n-1) + a\beta\pi & \text{if } n \text{ even}. \end{cases}$$

Observe the first $n \neq 0$, such that $x_a^{\alpha,\beta}(n) = x_a^{\alpha,\beta}(0) = \theta$.

For construction of the parameter a, we have

$$x^{\alpha,\beta} = \theta$$

$$a^{\alpha,\beta}(k,n) := \begin{cases} \frac{4k}{(\alpha+\beta)(n-1)+2\alpha} & \text{if } n \text{ odd,} \\ \frac{4k}{(\alpha+\beta)n} & \text{if } n \text{ even.} \end{cases}$$

- For any real numbers $a, \alpha, \beta \in \mathbb{R}$ provided that $a\alpha, a\beta \in \mathbb{Q}^+$.
- This study special cases for the choices of the parameters could lead to interesting results?

Generating classical geometric shapes

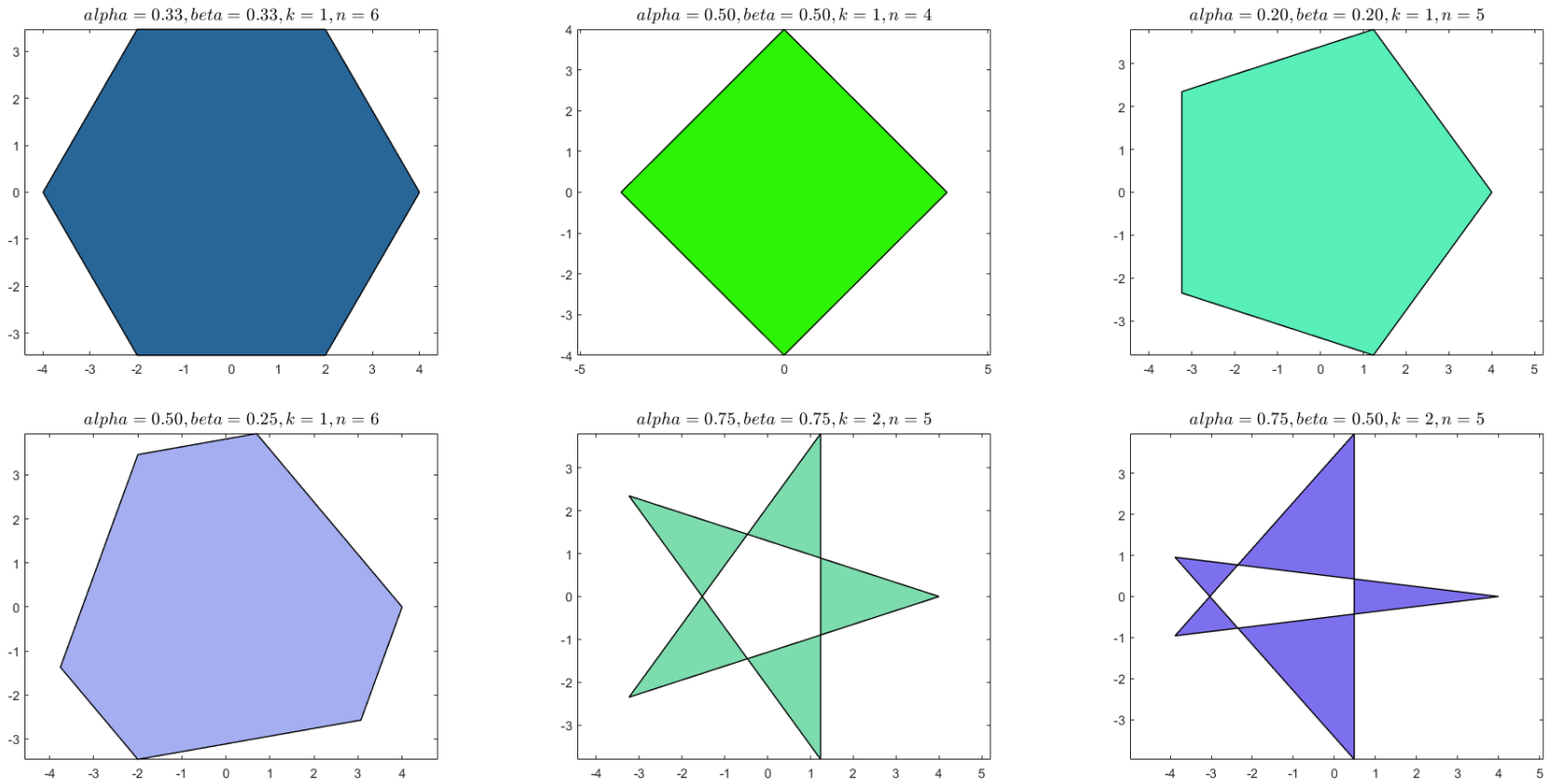


FIGURE: Cells for various parameters α, β, k and k.

Generating geometric shapes using large 'even' numbers

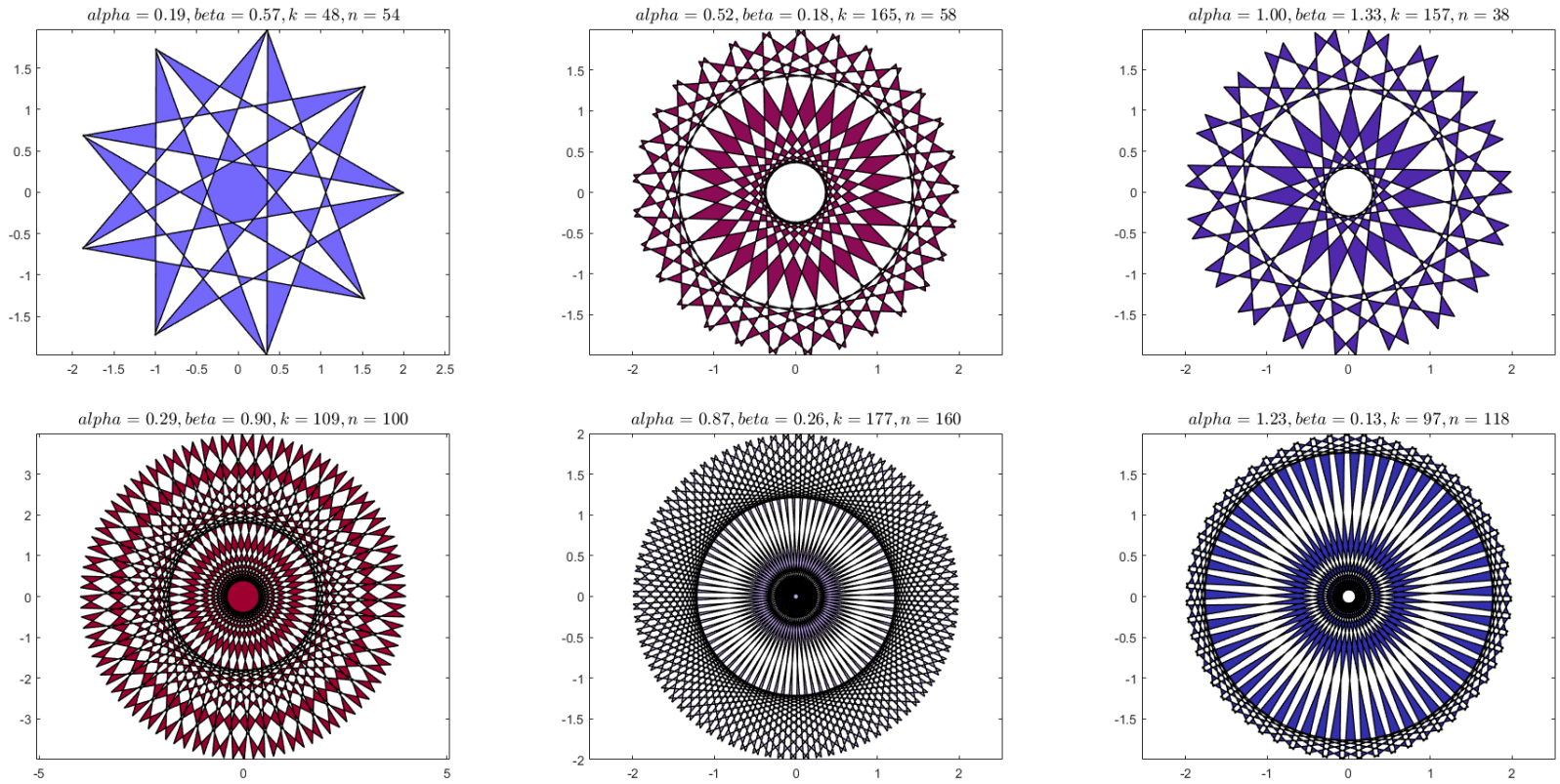


FIGURE: Cells for various parameters α, β, k and n.

Generating geometric shapes using large 'odd' numbers

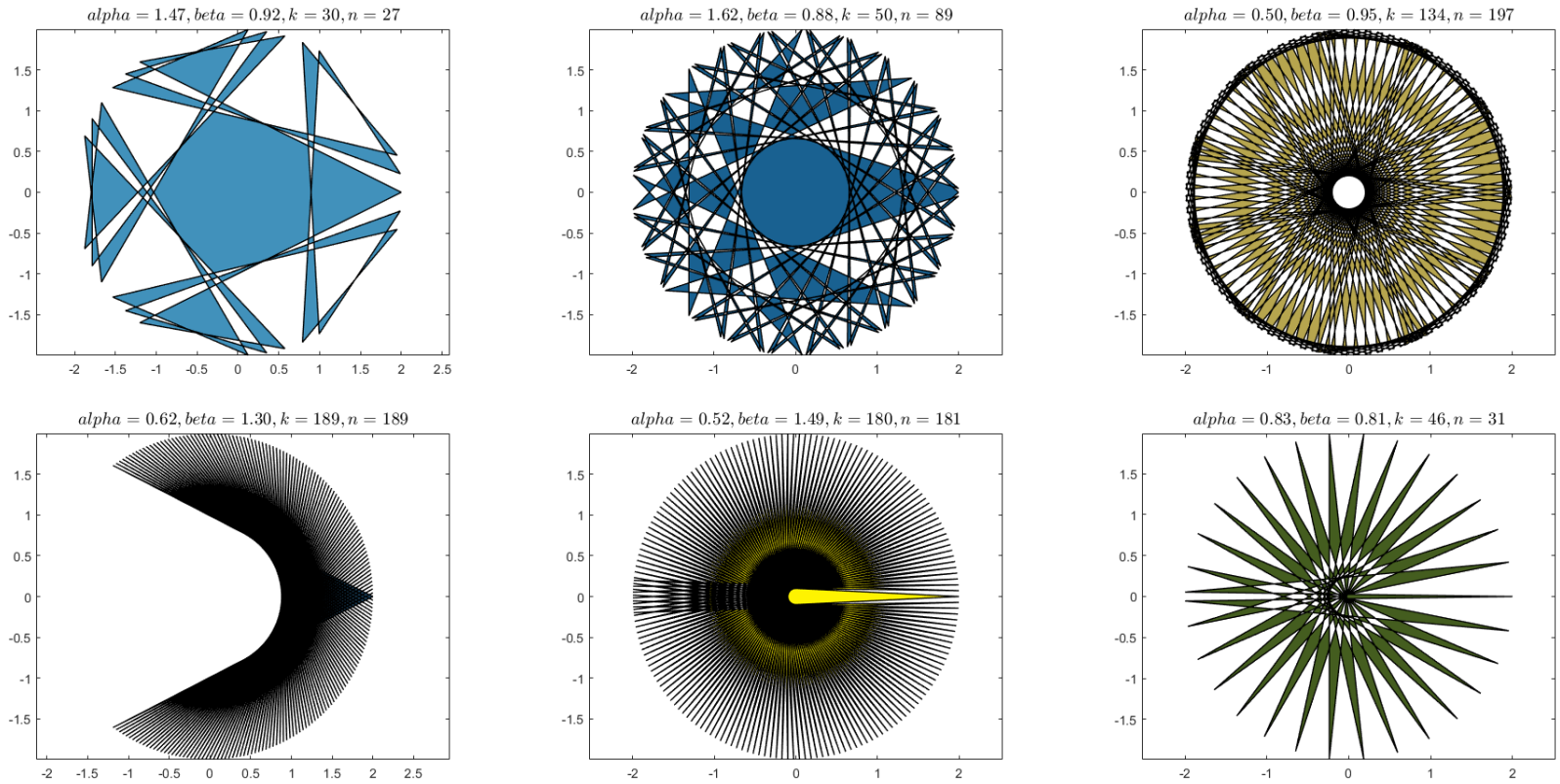


FIGURE: Cells for various parameters α, β, k and n.

For any initial value $x(0) = \theta$, we iteratively define the sequence

$$x_a : \mathbb{N} \longrightarrow S^1 \mod 2\pi$$

$$x_a(0) = \theta$$
$$x_a(n) := x_a(n-1) + a\pi$$

- Define $R(n) \subset \mathbb{R}^+$ for $n \in \mathbb{N}$ as a **Layer sequence** (any sequence having positive outputs).
- Instead of considering $x_a(n)$ on S^1, we consider $R(n)x_a(n)$ on S^p, the natural number p represent the cardinal of the following set

$$R := \{x \in \mathbb{R} \mid x = R(n) \quad \text{for} \quad n_f\}$$

- $x_a(n_f) = x_a(0) = \theta$.

For construction of the parameter a, we have

$$x^{\alpha,\beta} = \theta$$

$$a(k,n) := \frac{2k}{n},$$

In this case we don't need to distinguish between od and even number of vertices.

CHOICE OF THE PARAMETER a: STOPPING CONDITION

For the layer sequence with support $R := \{1, 1, 4, 5, 4, 1, 1\}$, we obtain

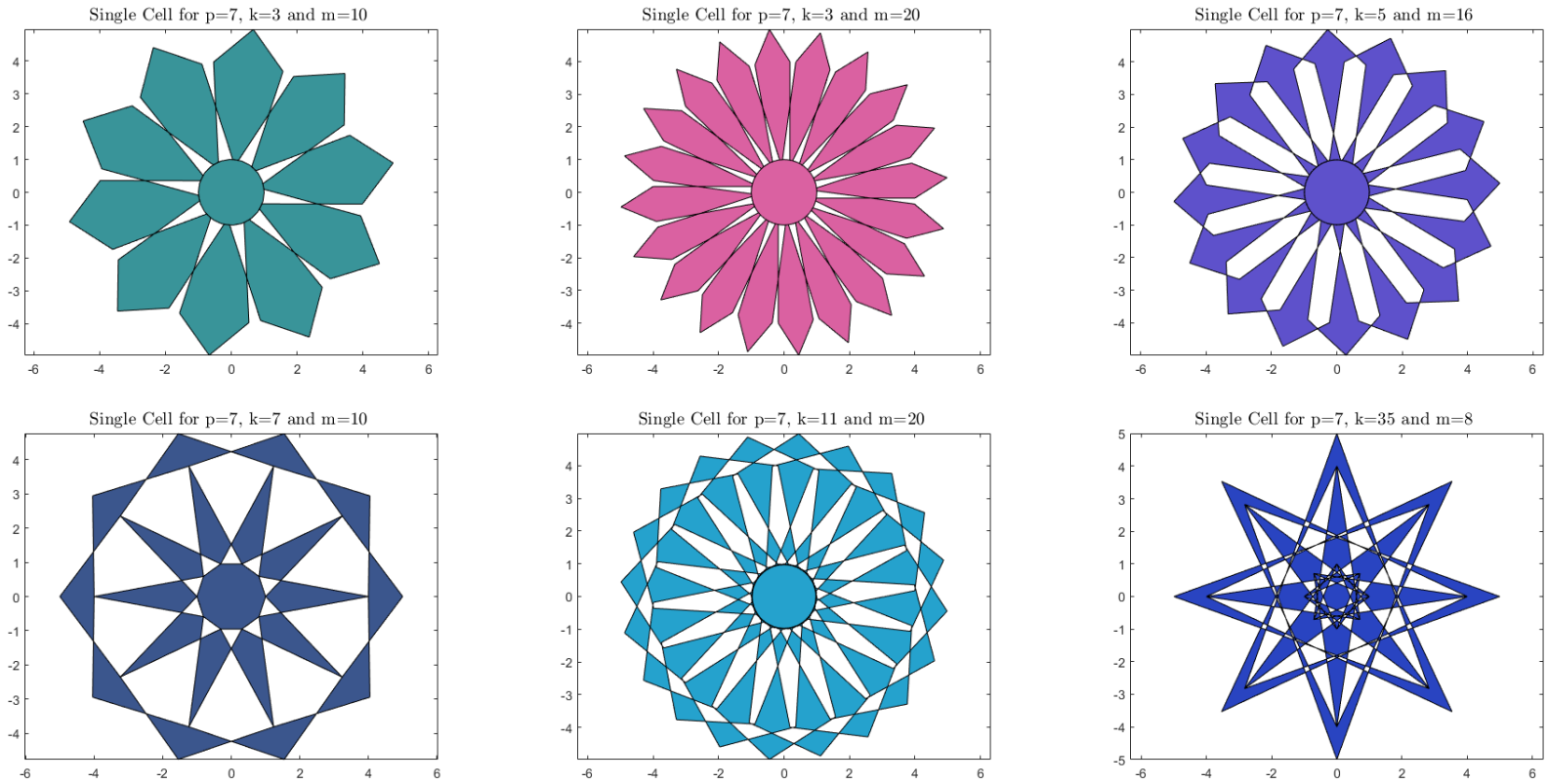


FIGURE: Cells for various parameters p, m and k.

Interpolating the vertices means adding more edges

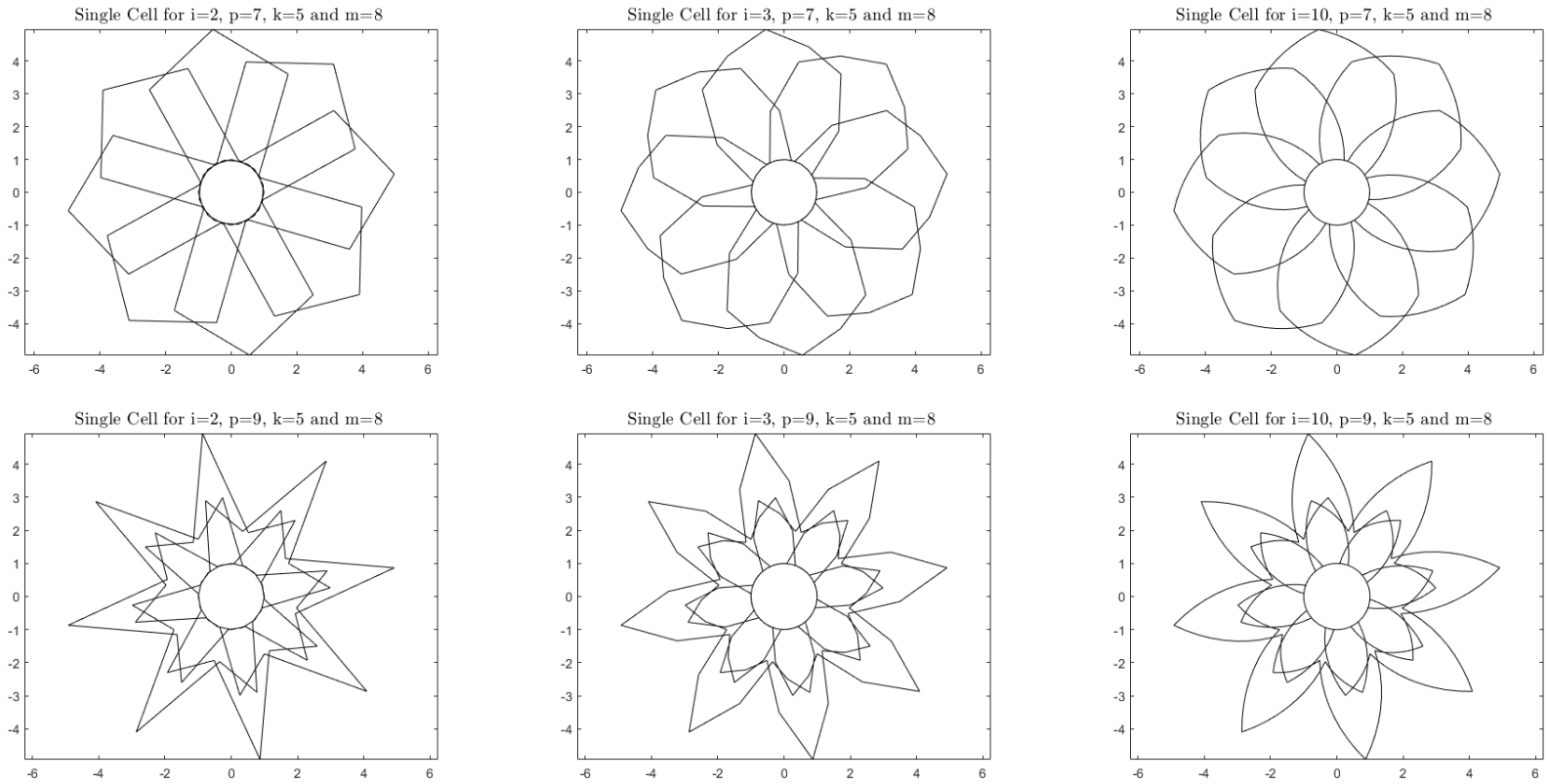


FIGURE: Interpolation effect on the shape.

INTERPOLATING VERTICES

For the layer sequence with support $R \subset \boldsymbol{N}$ and interpolating the vertices, we obtain

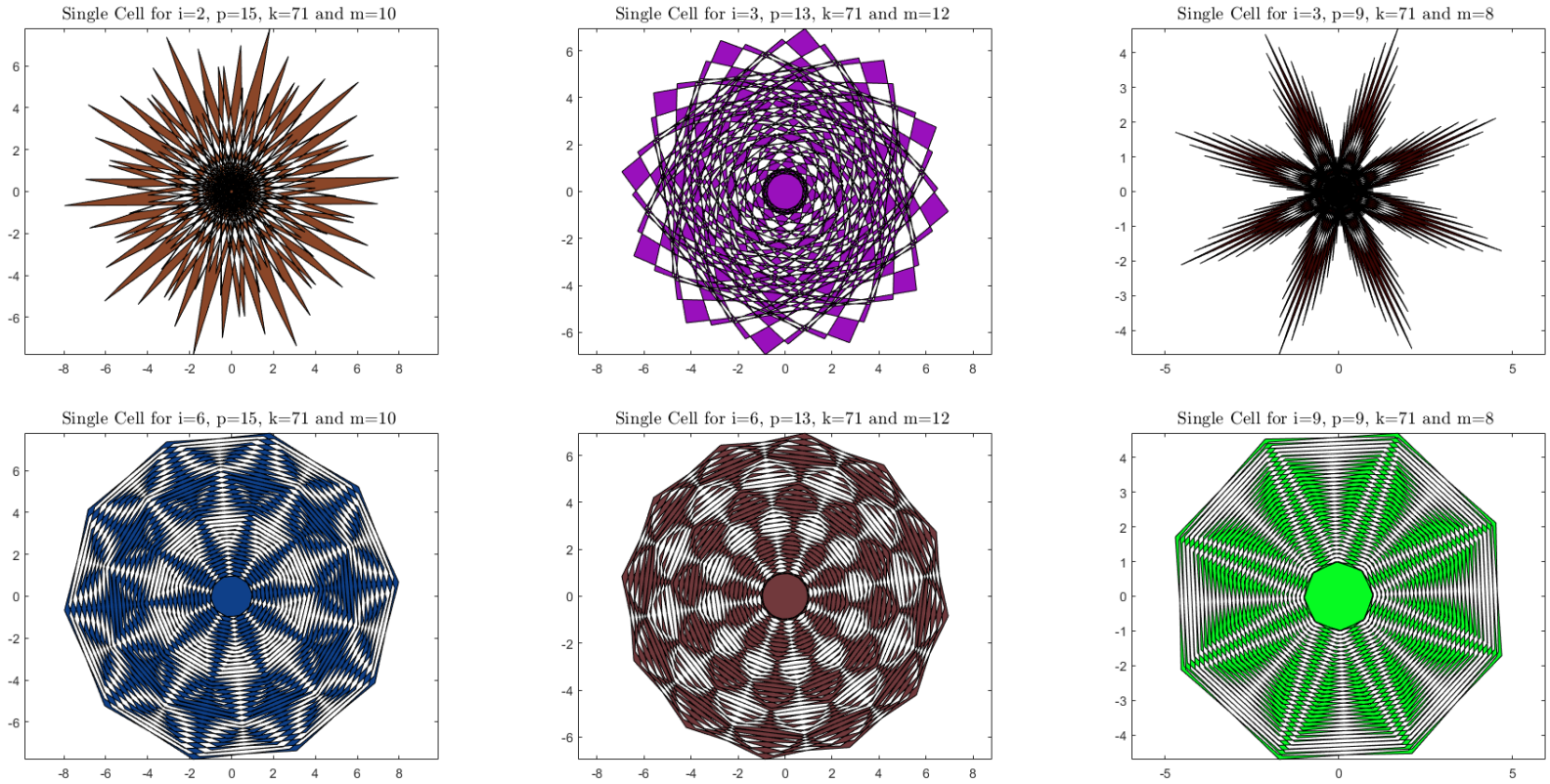
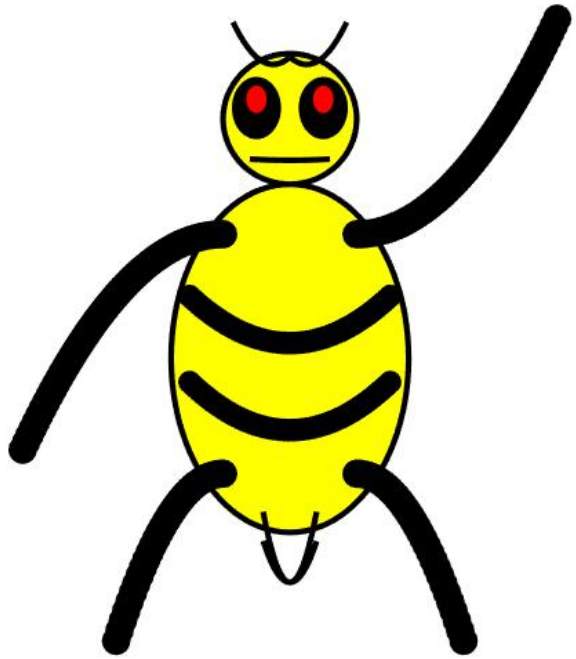
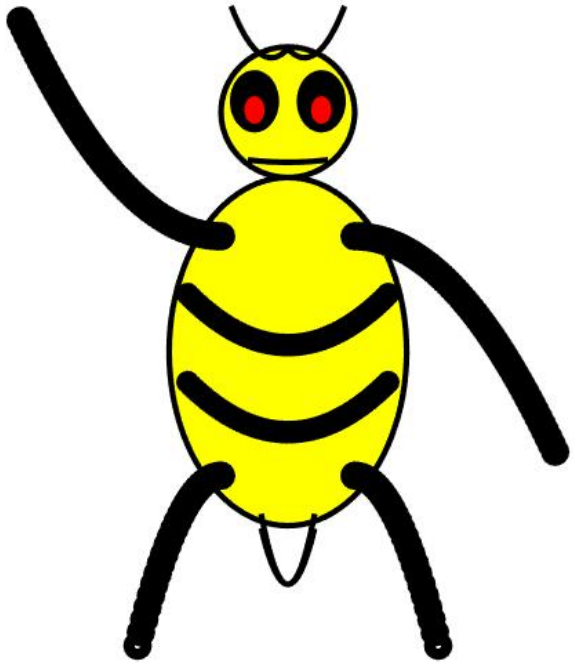
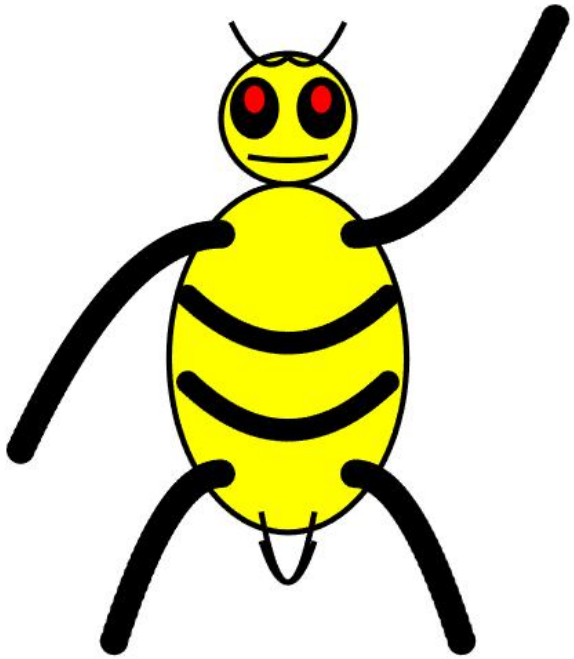


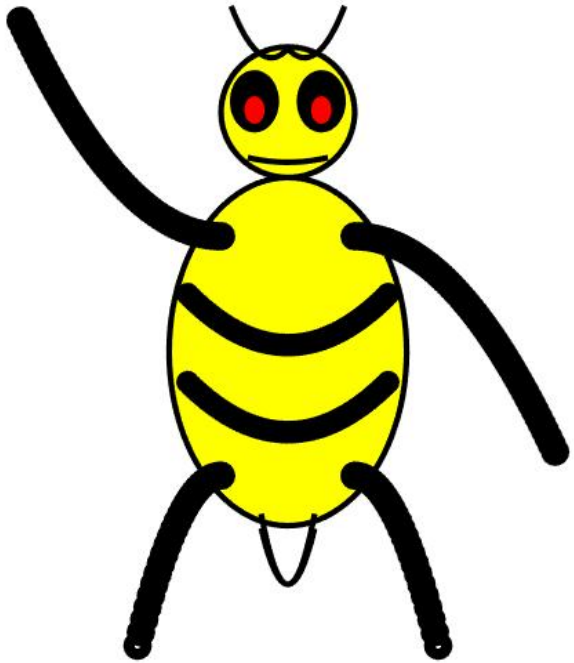
FIGURE: Cells for various parameters p, m and k.

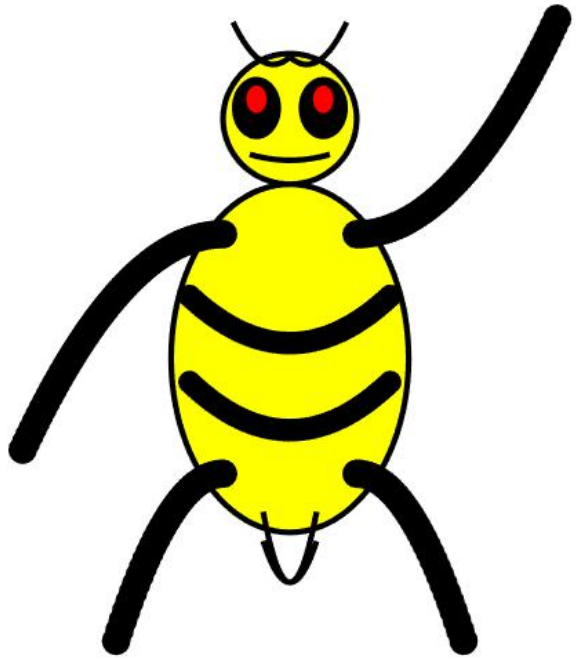


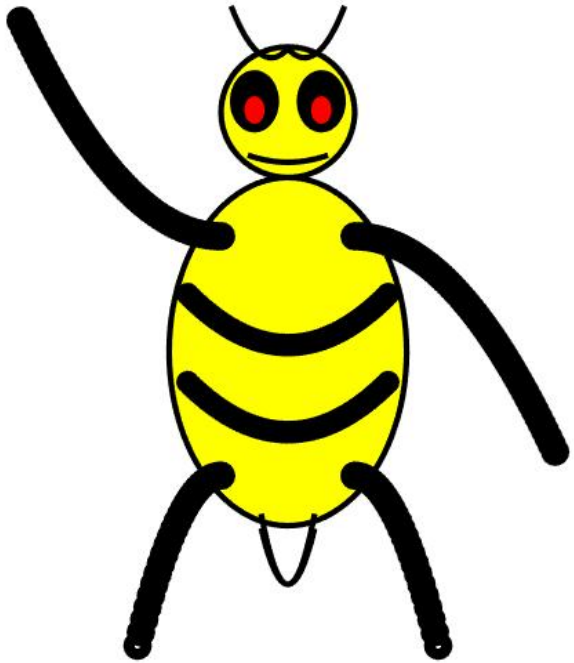


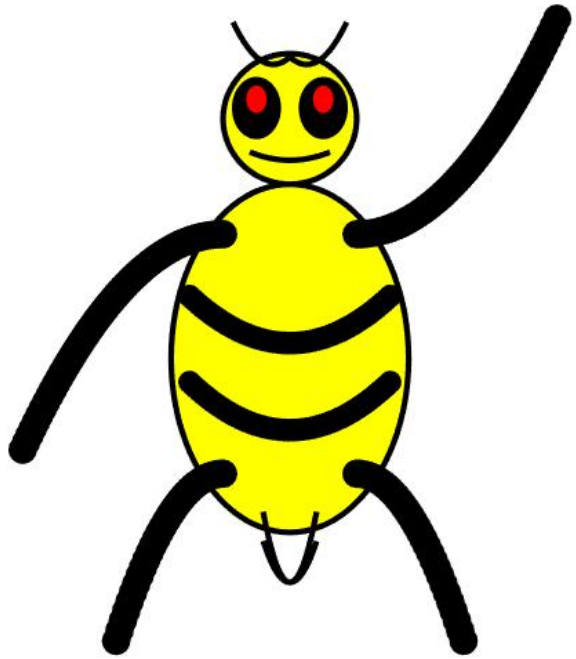
Before the second part: A bit of stretching

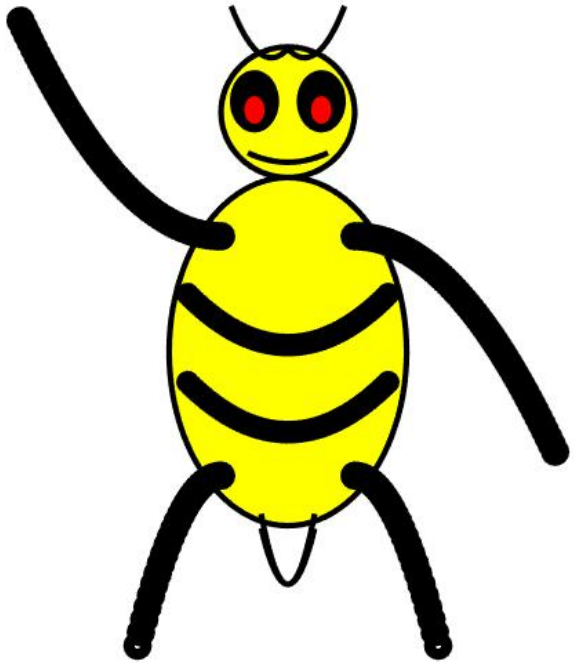


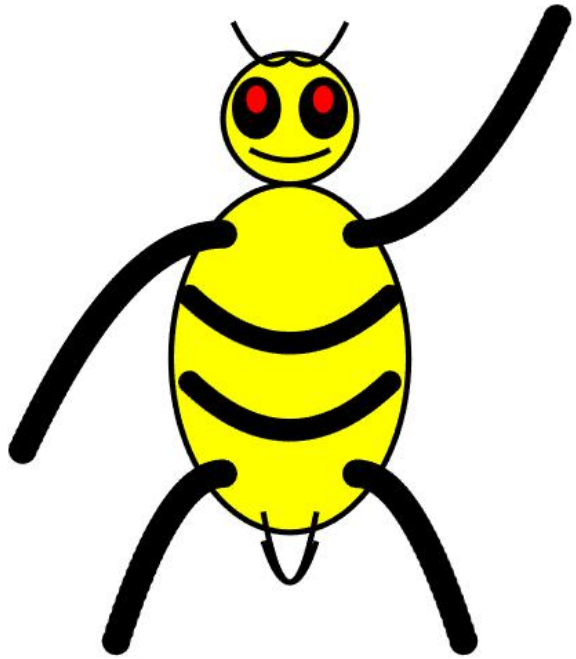


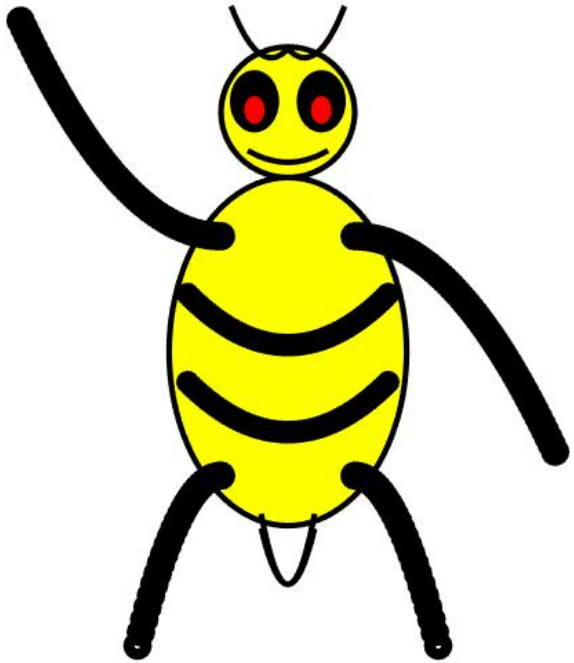




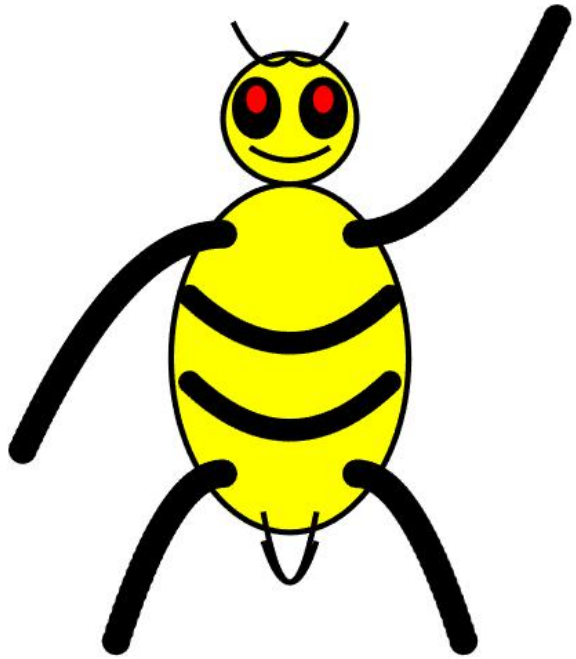


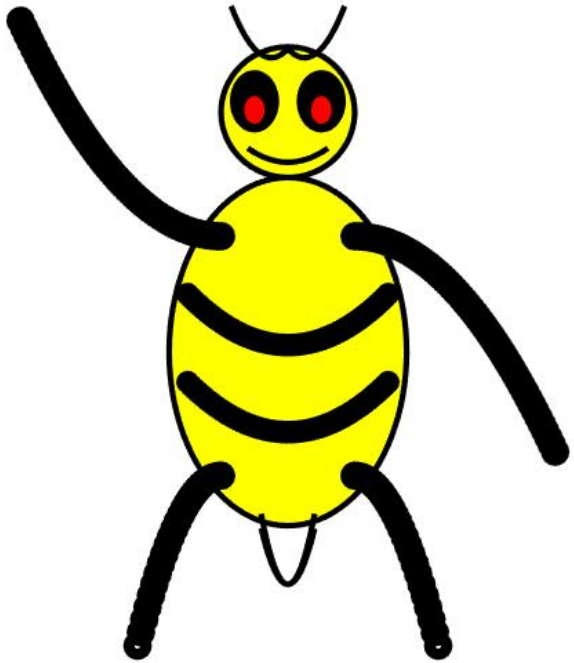




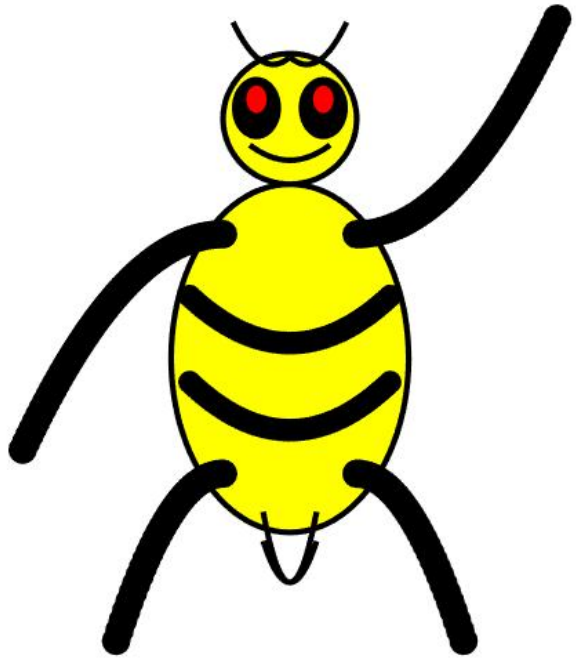


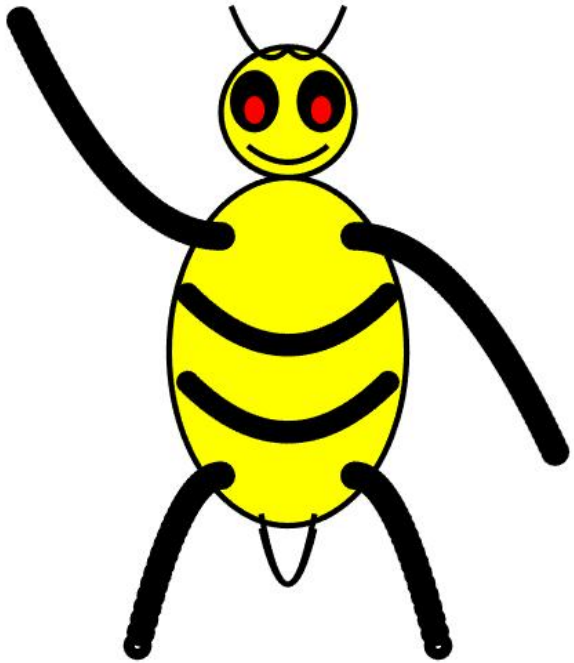
Before the second part: A bit of stretching

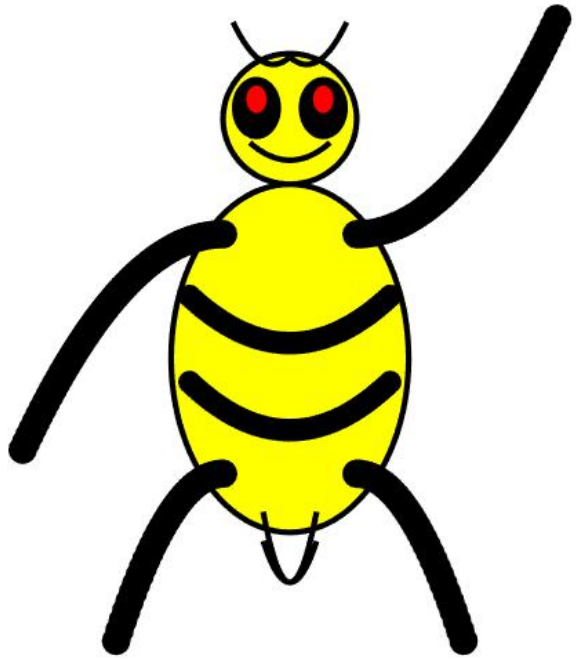




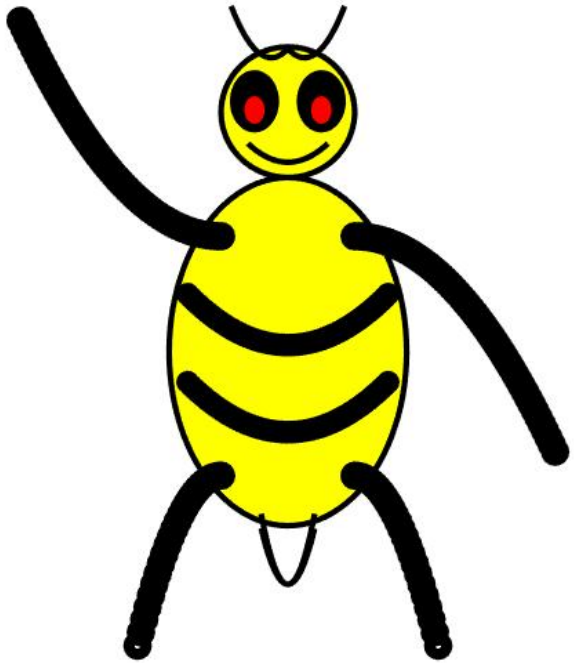
Before the second part: A bit of stretching

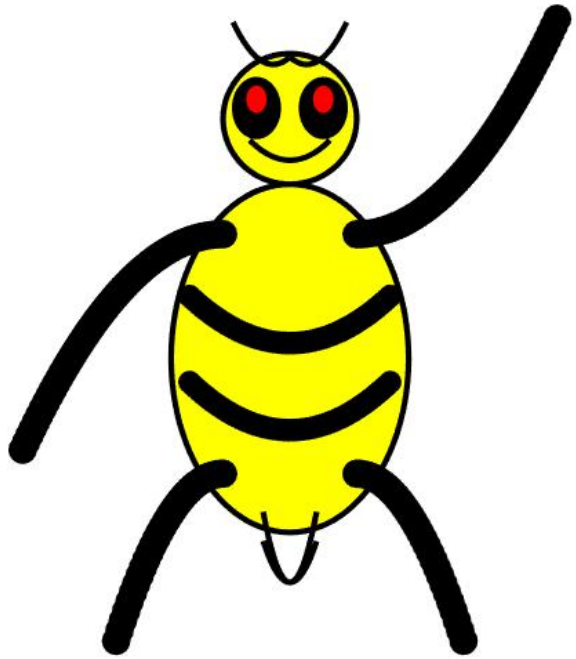


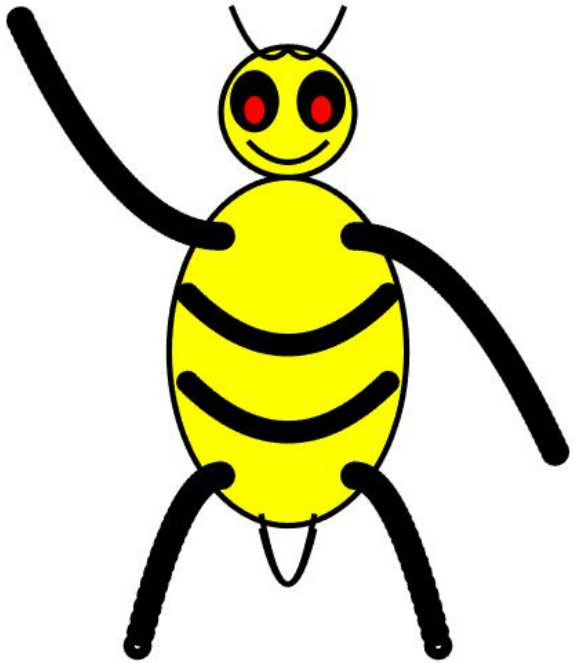


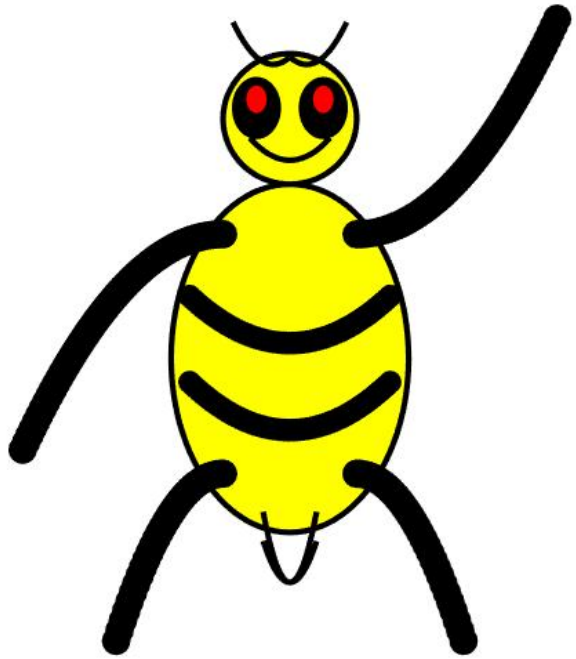


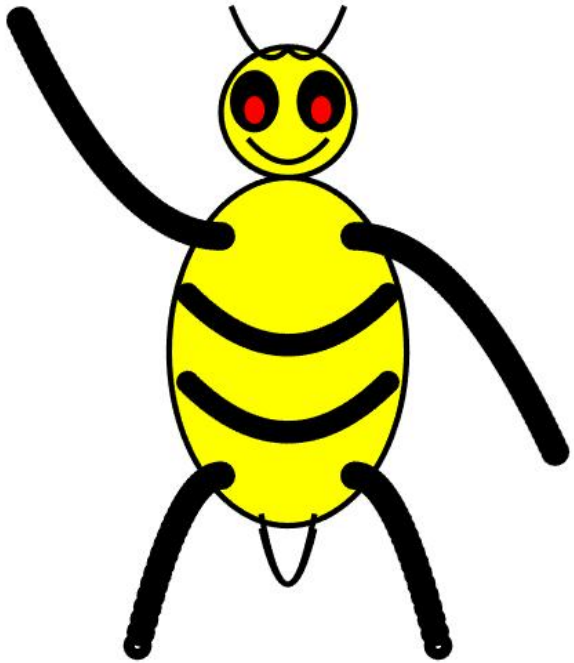
Before the second part: A bit of stretching



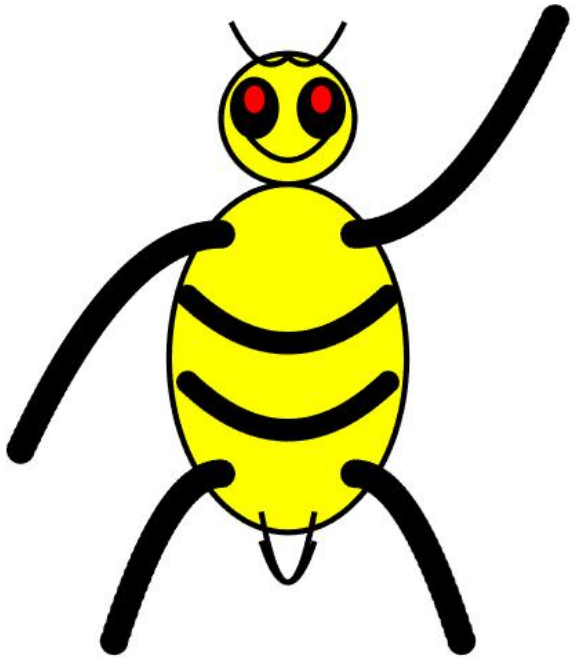








Before the second part: A bit of stretching



Space filling techniques: Linear function

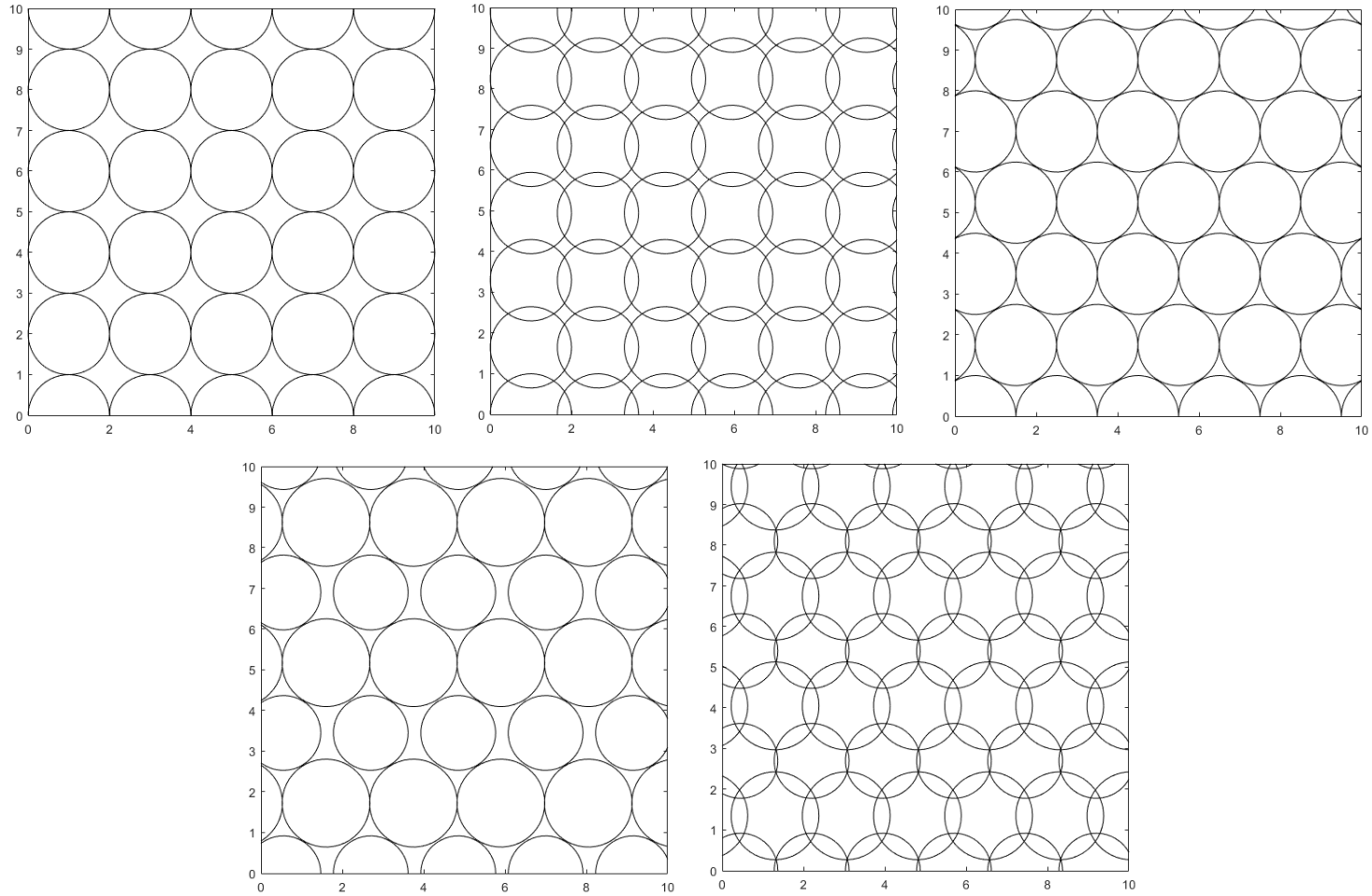


FIGURE: Space filling techniques: Linear forn

Space filling techniques: Non-linear function

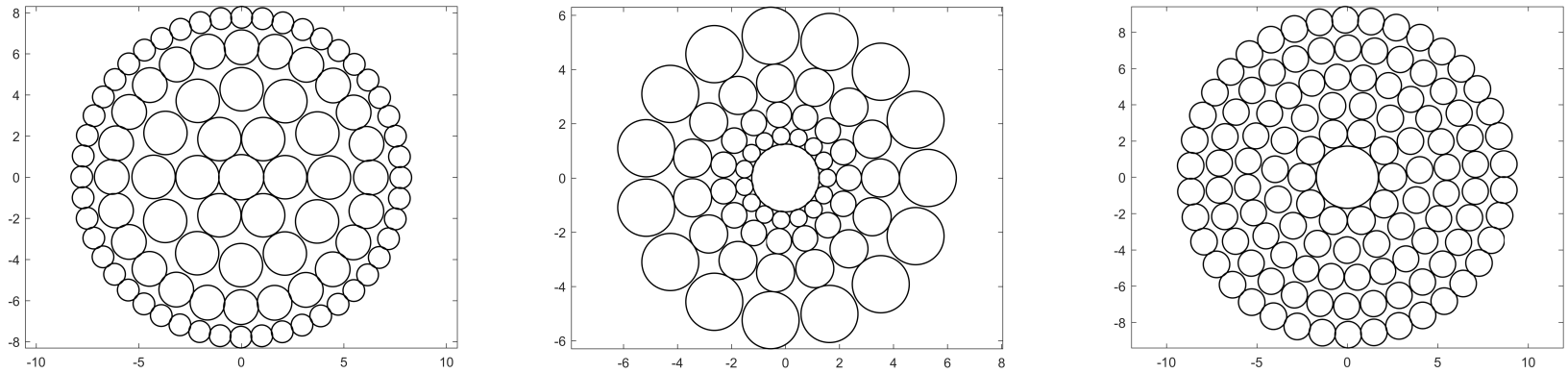
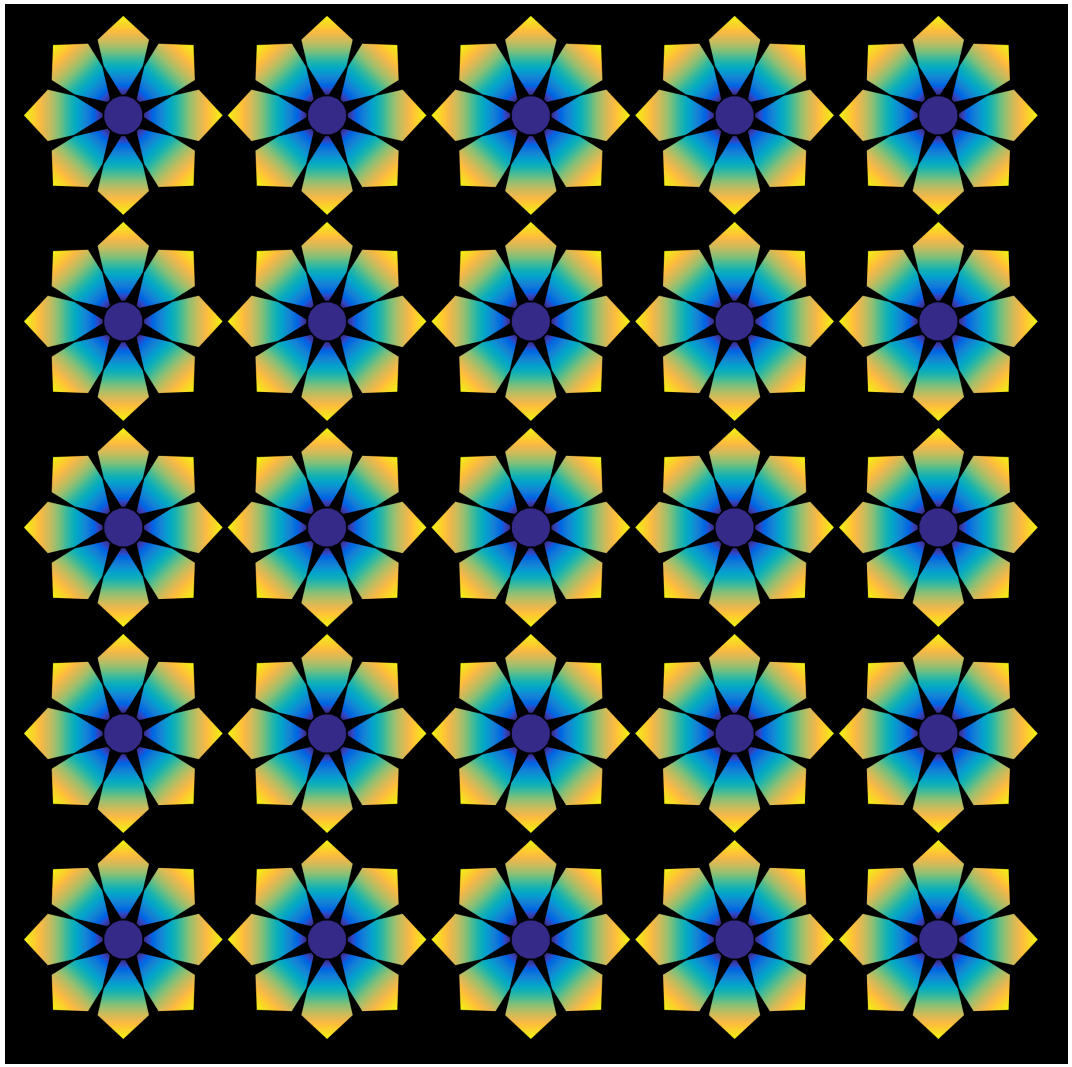
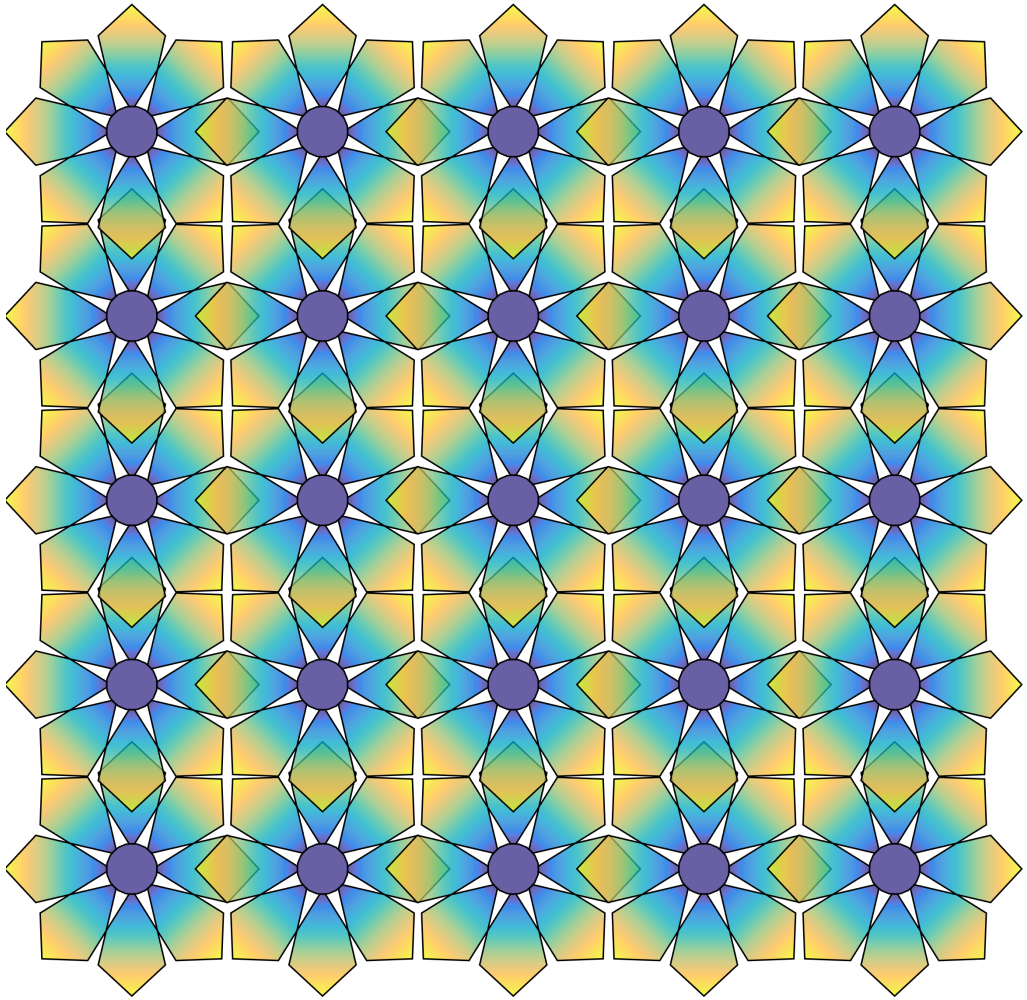


FIGURE: Space filling techniques: Circular Forms

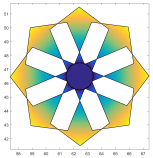
Linear Tessellation of similar patterns

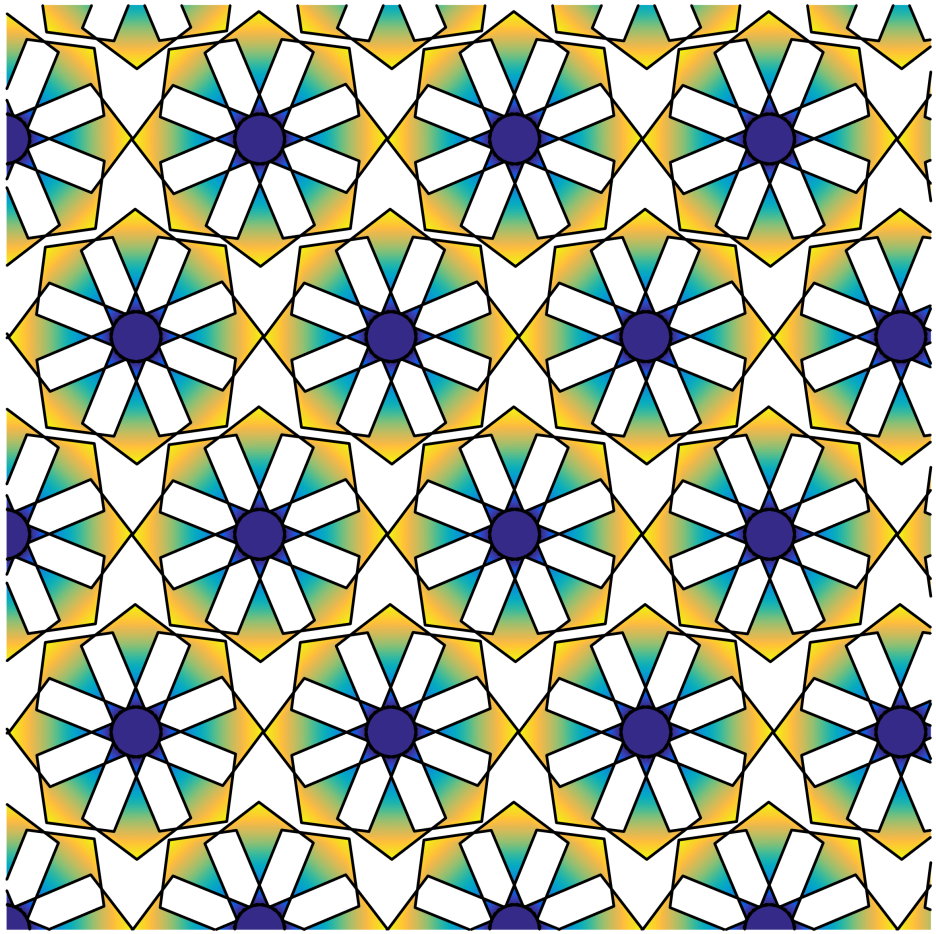


Linear Overlapped patterns

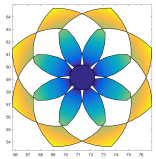


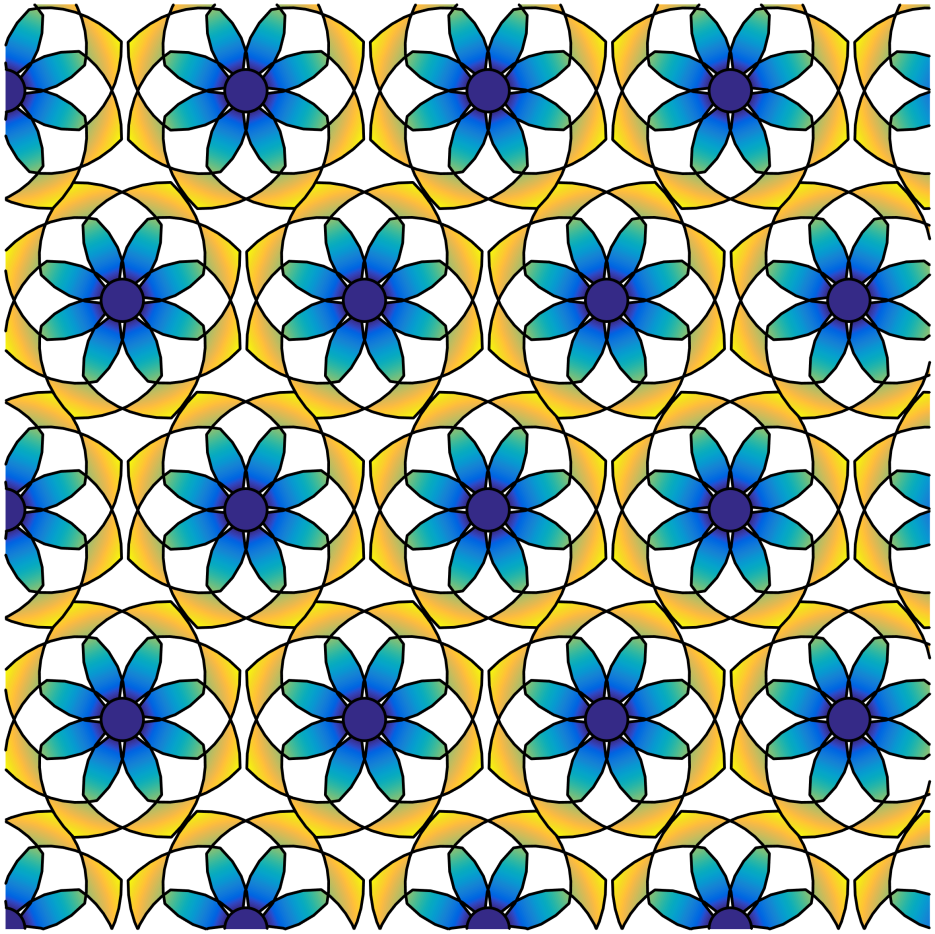
Planar Tessellation of 8-Folds Moroccan Zalij with linear edges



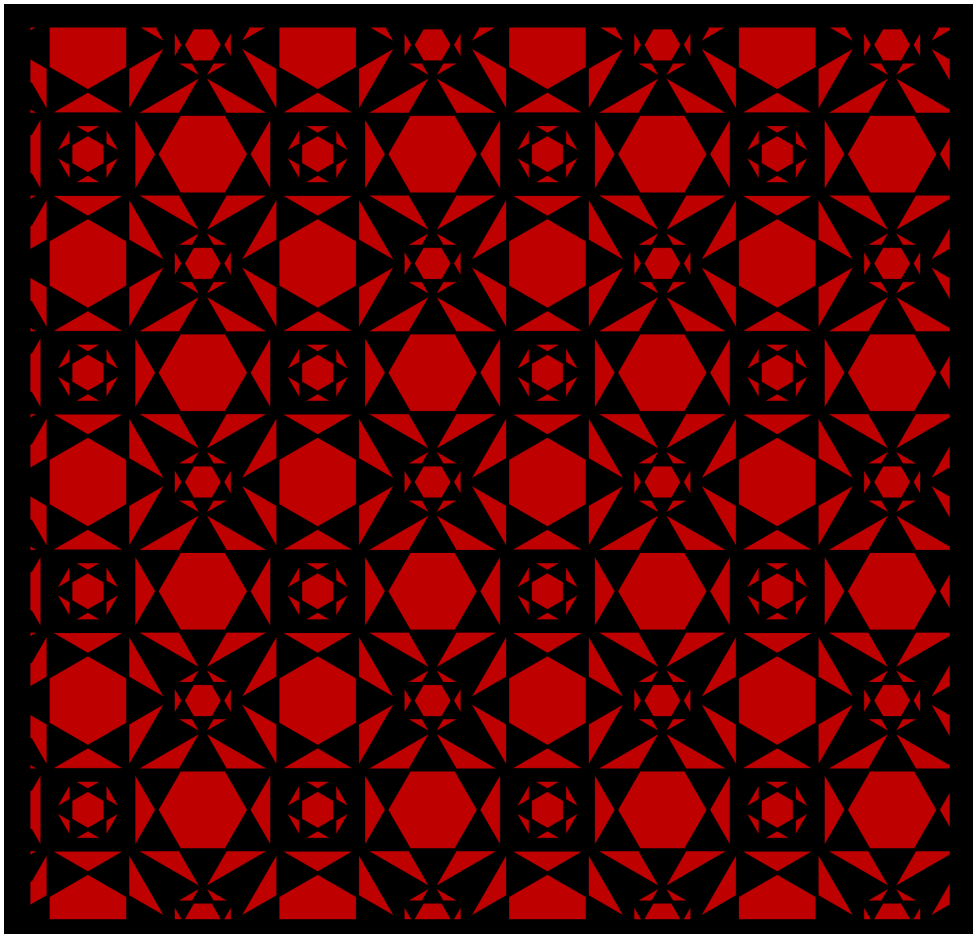


Planar Tessellation of 8-Folds Moroccan Zalij with non-linear edges

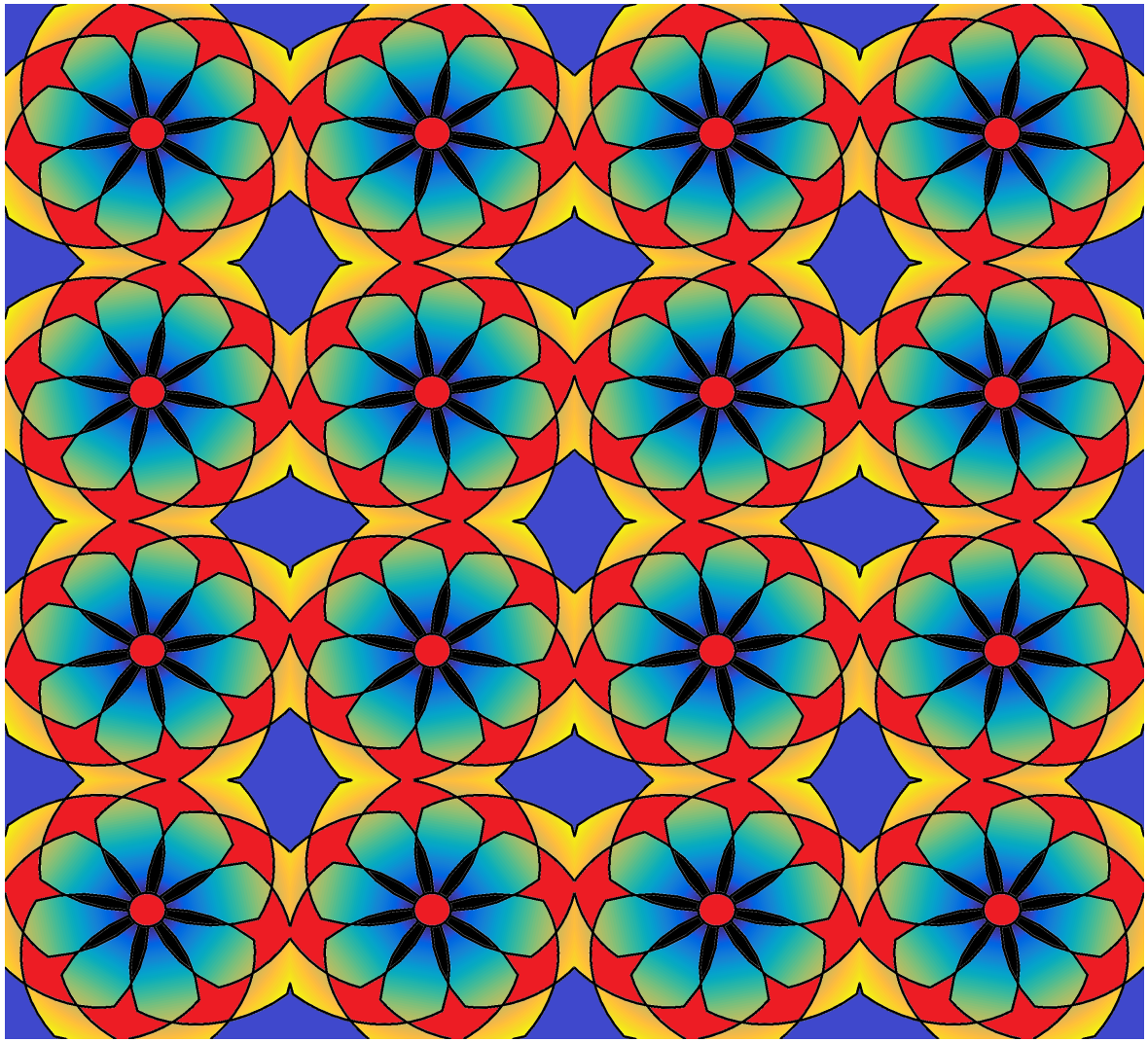




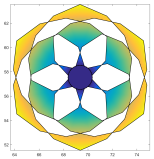
Linear Non-Overlapped patterns with different sizes

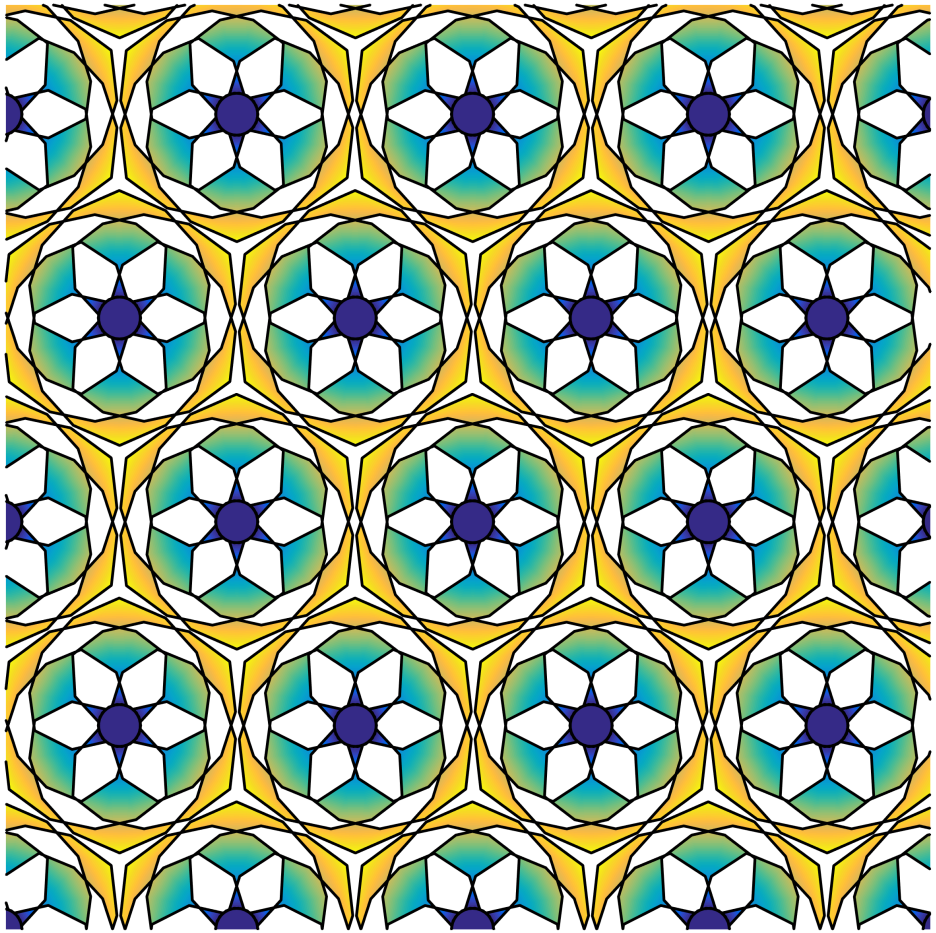


No-Linear Non-Overlapped patterns with symmetrical cross sections

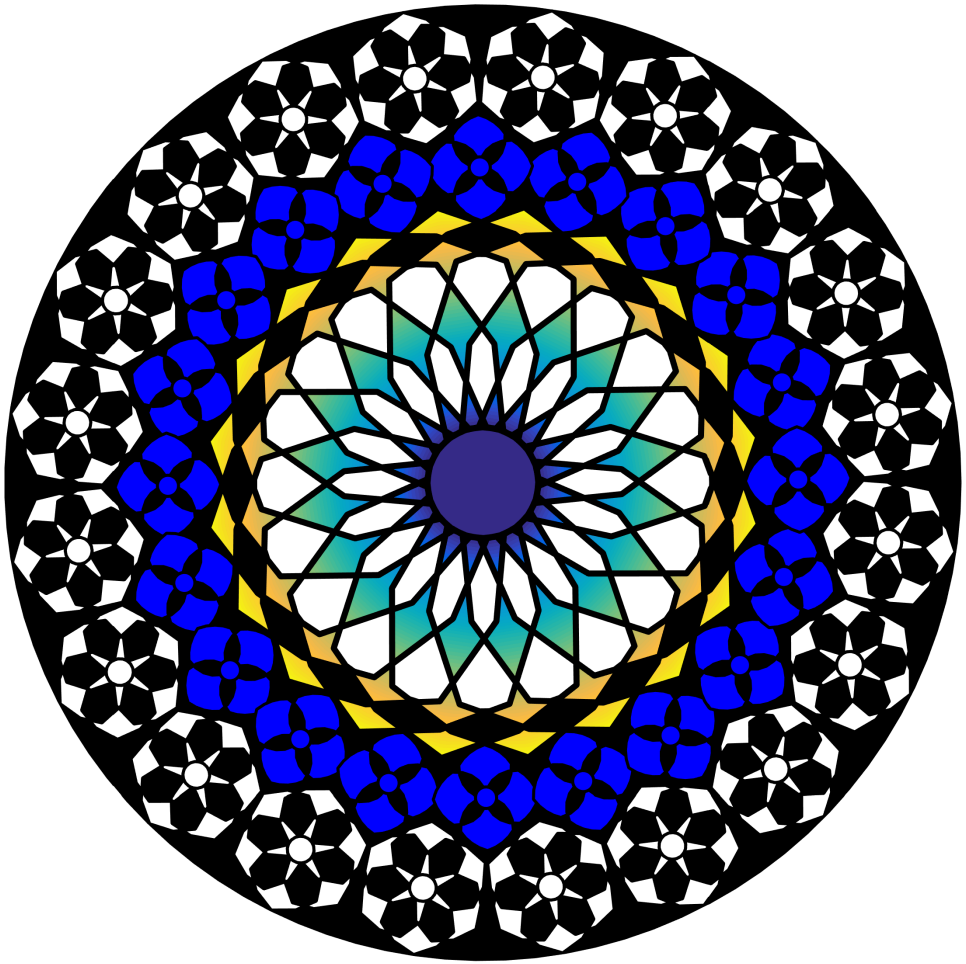


Planar Tessellation of 6-Folds Moroccan Zalij with Multi-linear edges

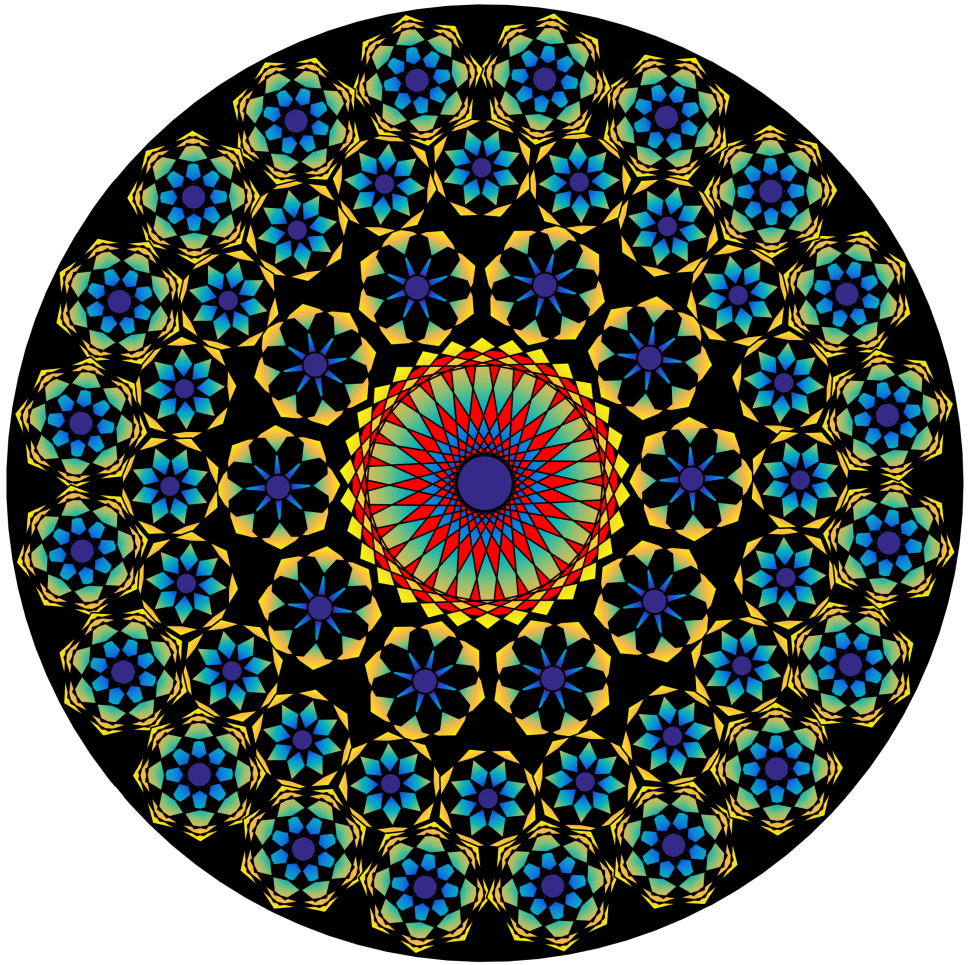




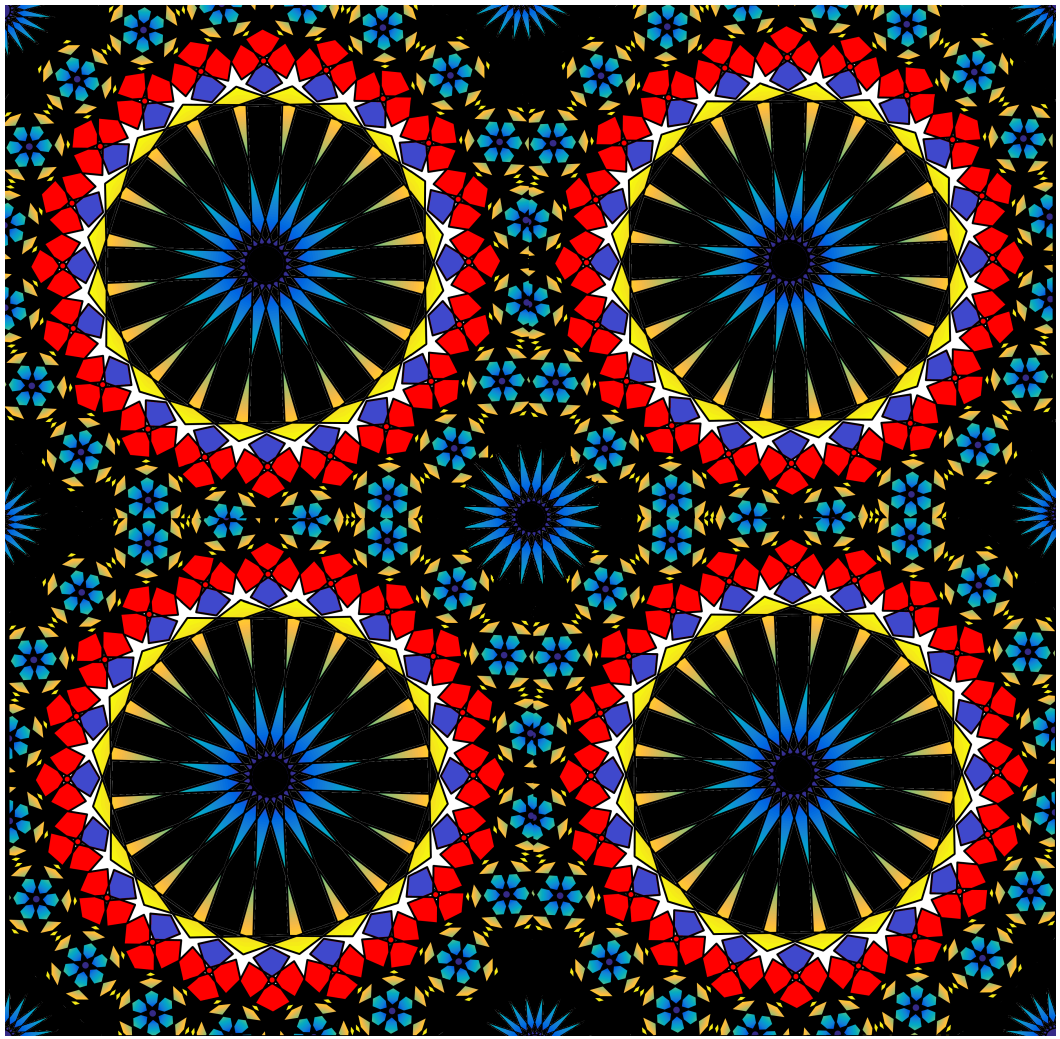
Circular Tessellation of Moroccan tiles with 3D effect



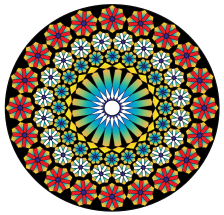
Mandala Design of a Moroccan Rosette kind

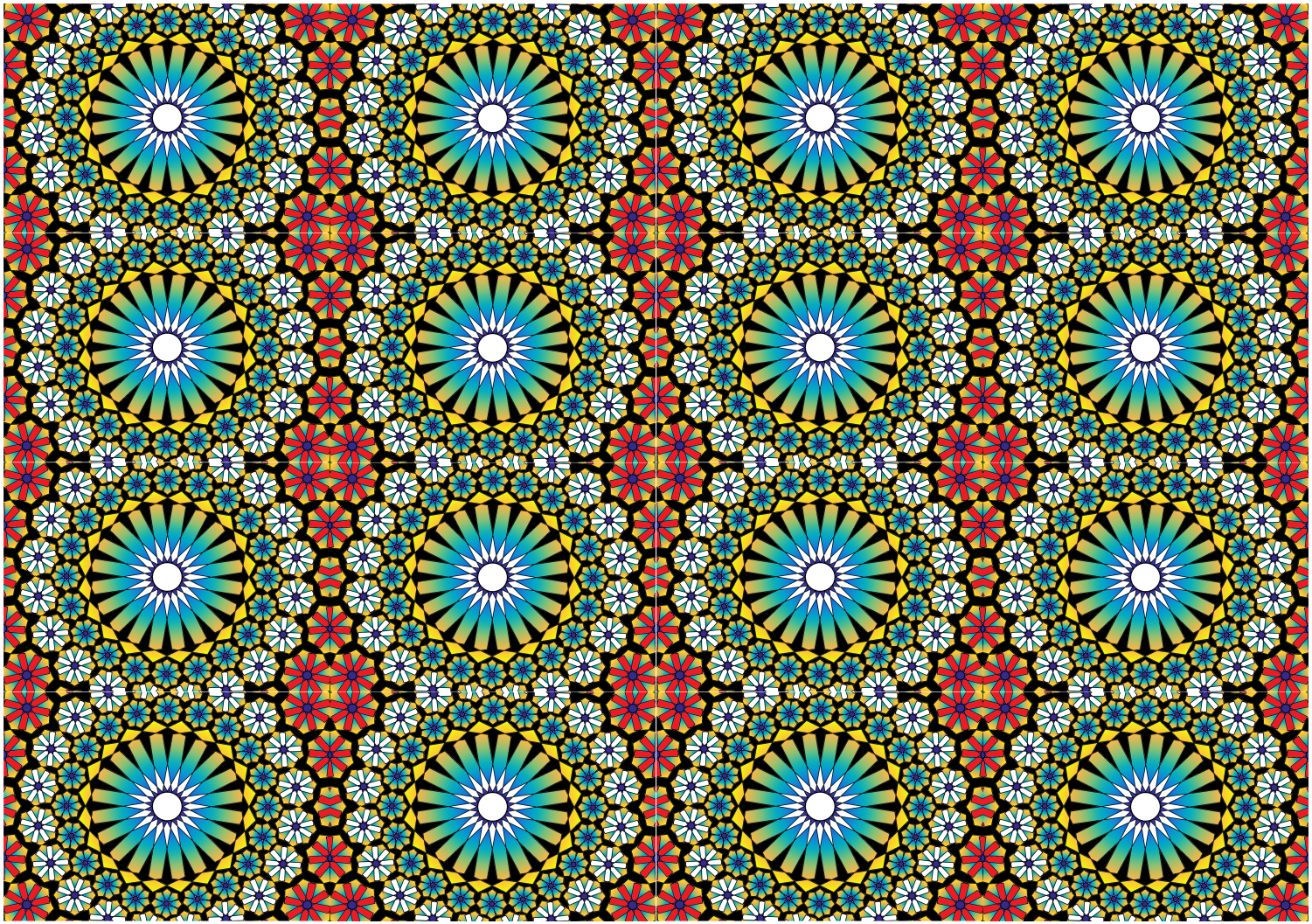


Planar Tessellation of Mandala Design with Moroccan Zalij

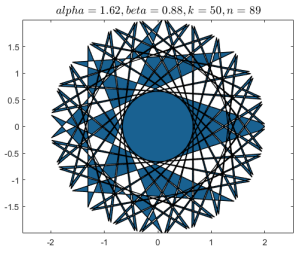


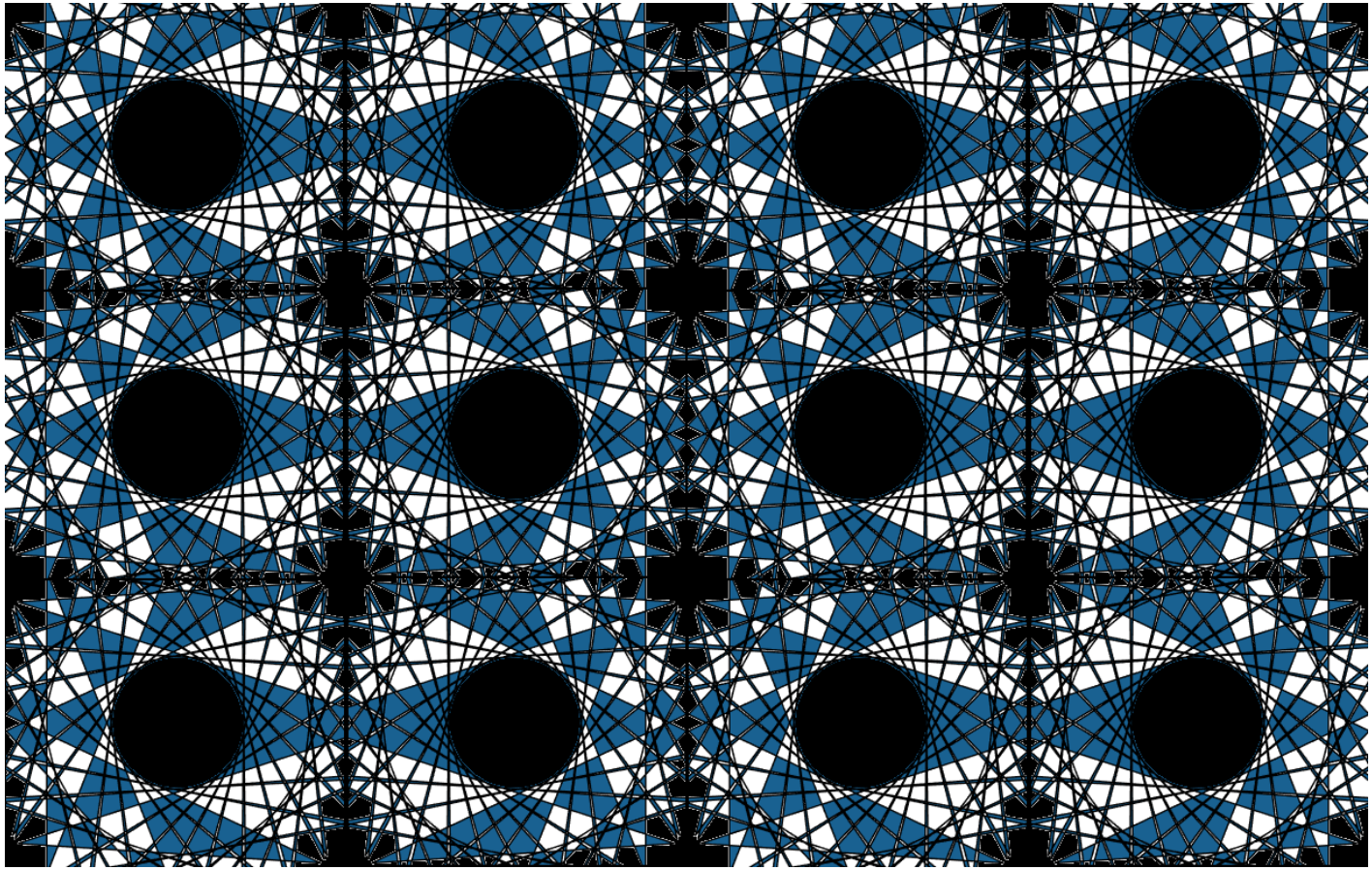
Design construction from multi-composite cells



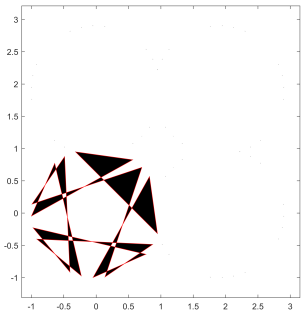


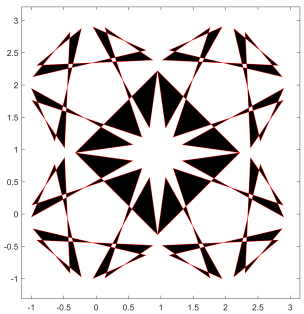
Design construction from multi-composite cells

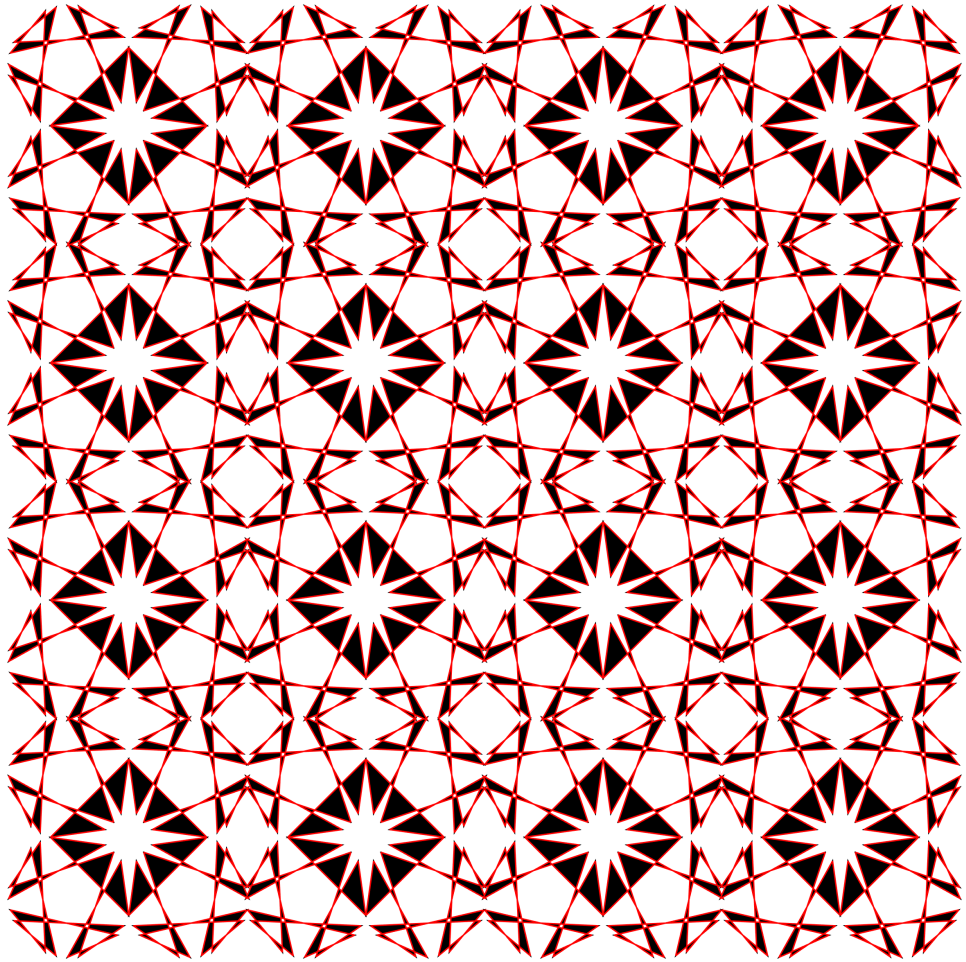




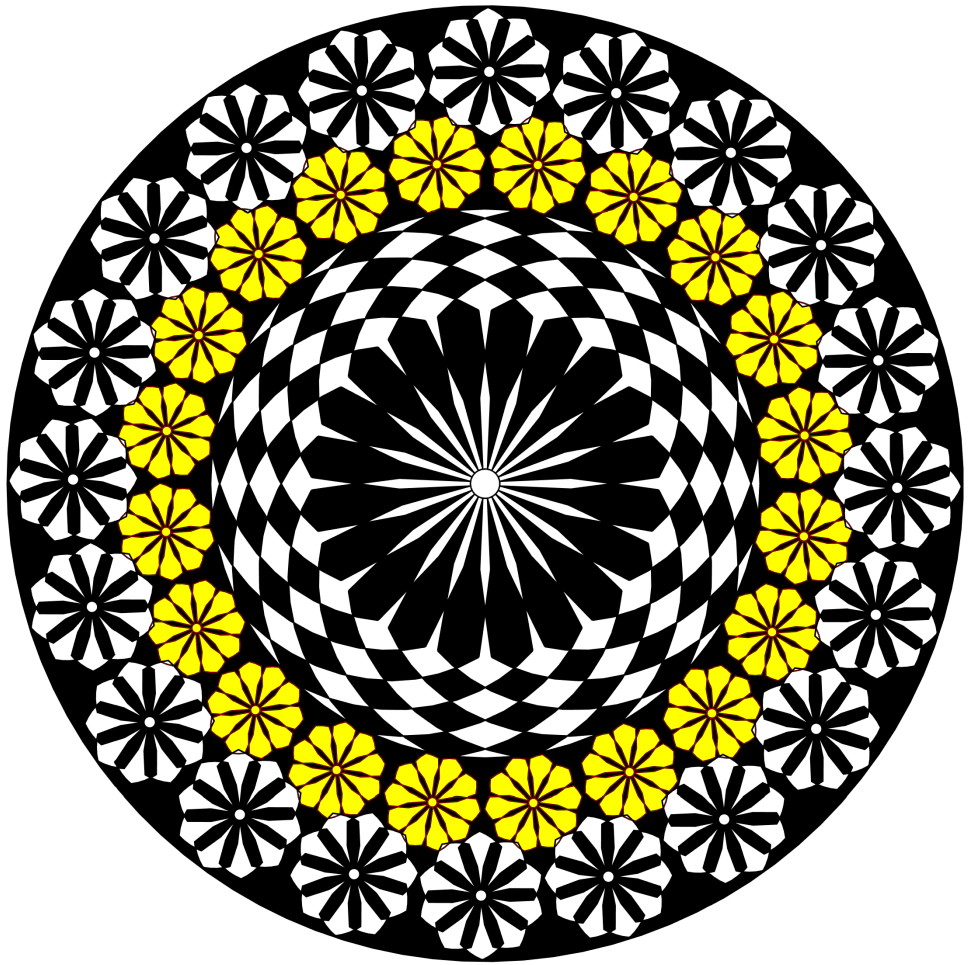
Oriented Tessellation of pseudo-symmetric cells for the construction of a unit cell







Mandala Design of a Moroccan Rosette kind

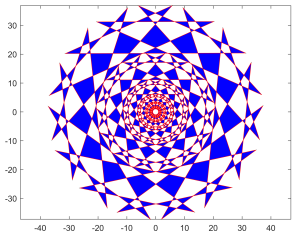


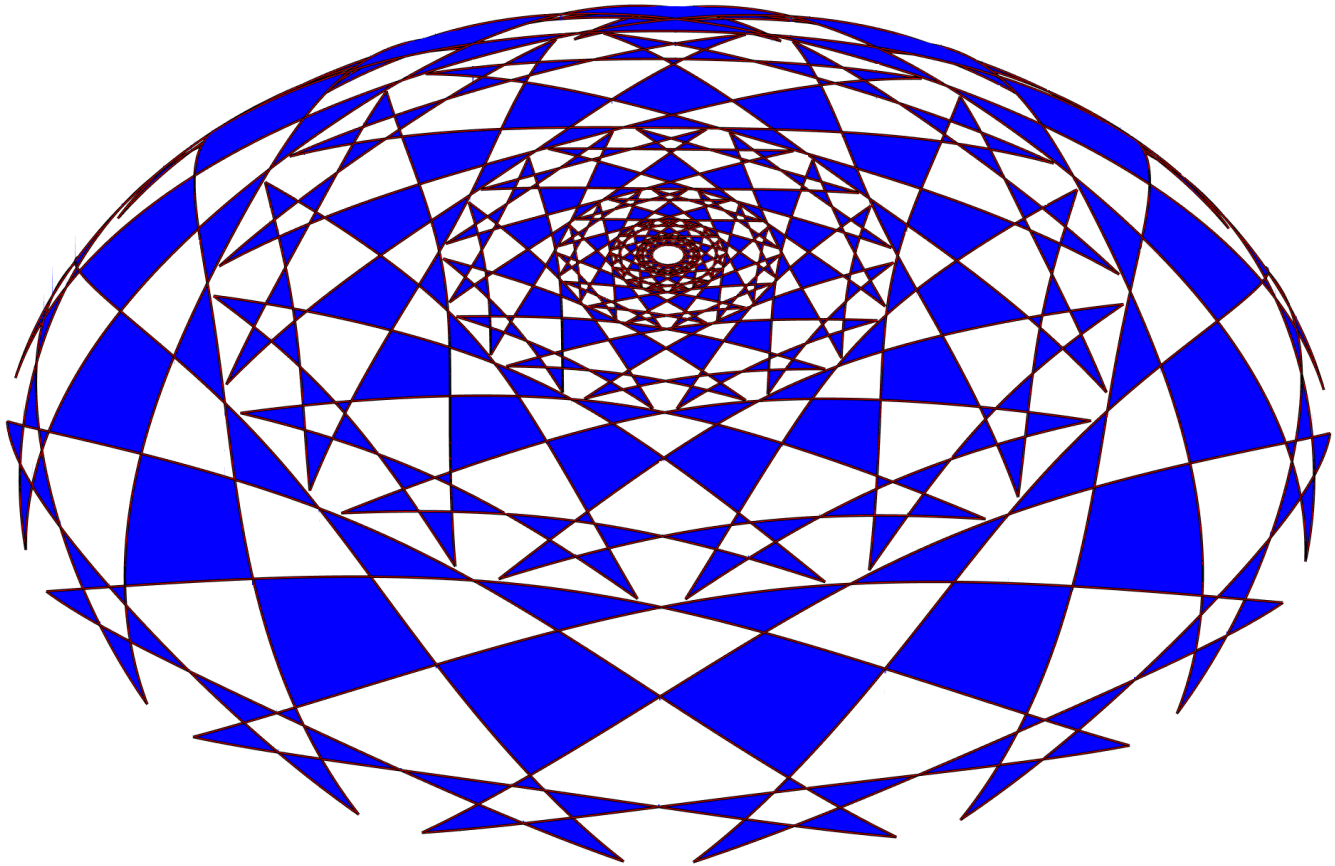
Given a disk $D(c,r)$ with centre $c=(x_d,y_d)$ and radius r. For any $r_s \geq r$, the following mapping is a stereographic projection of the planar Disc $D(c,r)$ into spherical points:

$$\begin{aligned} S_{r_s} : D(c,r) &\longrightarrow S^+(c_0,r_s) \\ (x,y) &\longmapsto \left(\frac{xr_s}{h}, \frac{yr_s}{h}, \frac{r_s(r_s-r_p)}{h}\right) \end{aligned} \tag{1}$$

where $S^+(c_0,r_s)$ is the half sphere for $z \geq 0$ with center $c_0=(0,0,0)$ and radius r_s such that $r_s^2 = x^2+y^2+z^2$ and $r_p = \left(x^2+y^2\right)^{\frac{1}{2}}$ represents the distance from the point (x,y) to the sphere centre $c=(x_s,y_s,z_s)$ and $h = \left(r_s^2 - 2r_p(r_s-r_p)\right)^{\frac{1}{2}}$.

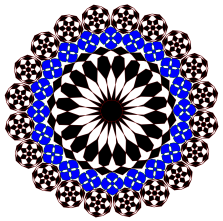
Three dimensional Rosettes(Stenographic projection)

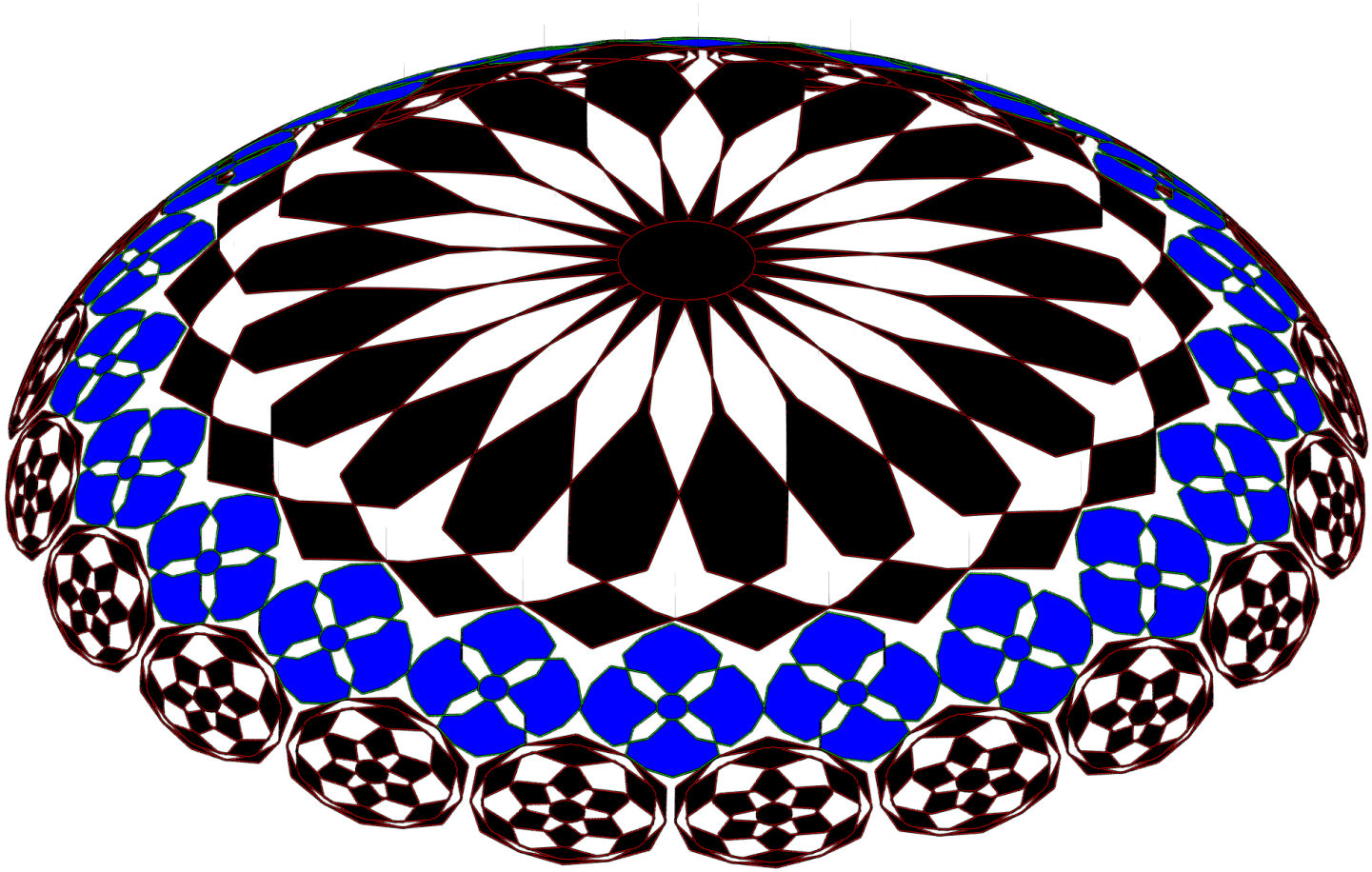




Compiling 3D Patterns

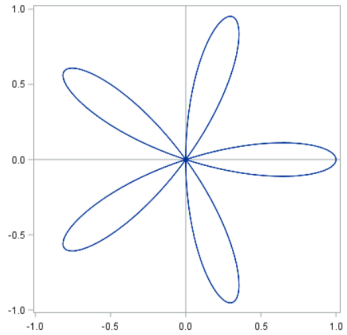
Three dimensional Rosettes(Stenographic projection)





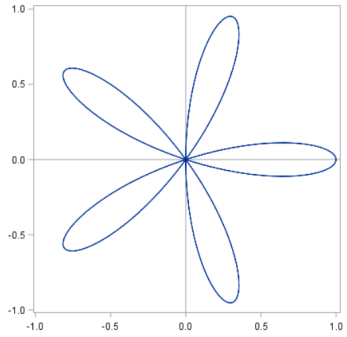
Simulation of Rose Equations

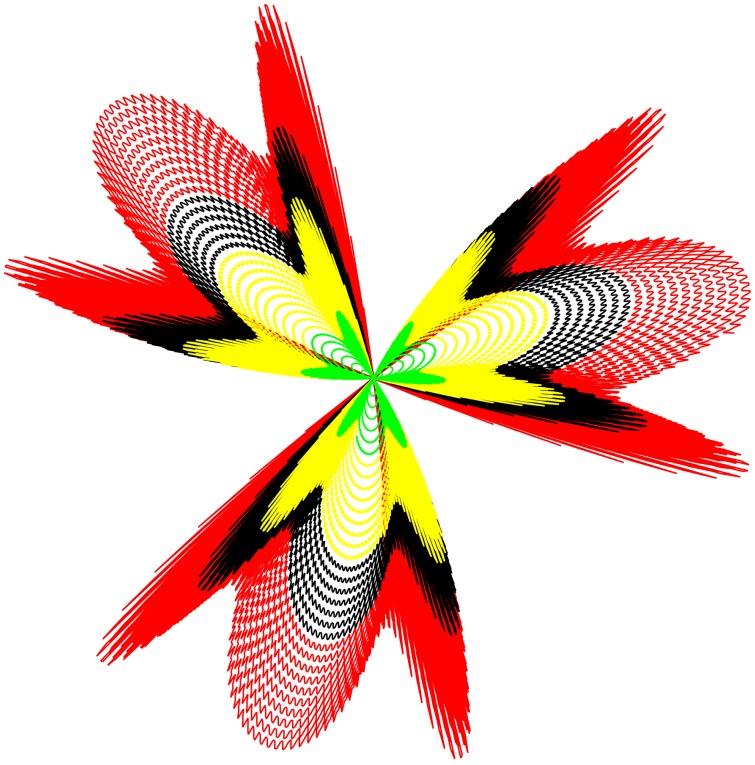
$$x(t) = A \cdot \cos(nt)\cos(t)$$
$$y(t) = A \cdot \cos(nt)\sin(t)$$



Simulation of Rose Equations

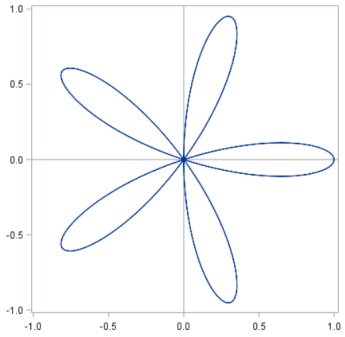
$$x(t) = A \cdot \cos(nt)\cos(t)$$
$$y(t) = A \cdot \cos(nt)\sin(t)$$





Simulation of Rose Equations

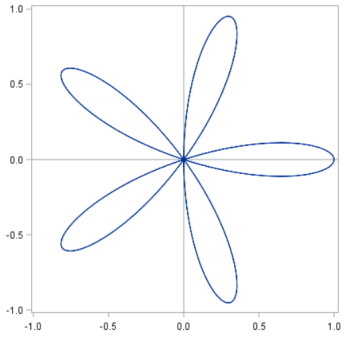
$$x(t) = A \cdot \cos(nt)\cos(t)$$
$$y(t) = A \cdot \cos(nt)\sin(t)$$

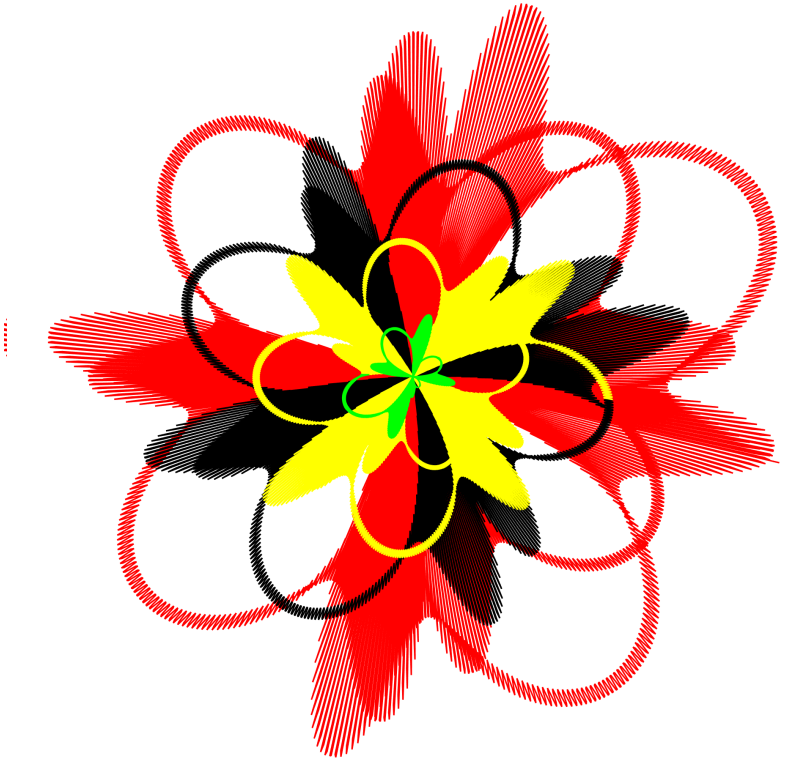




Simulation of Rose Equations

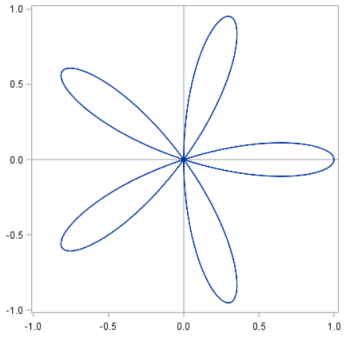
$$x(t) = A \cdot \cos(nt)\cos(t)$$
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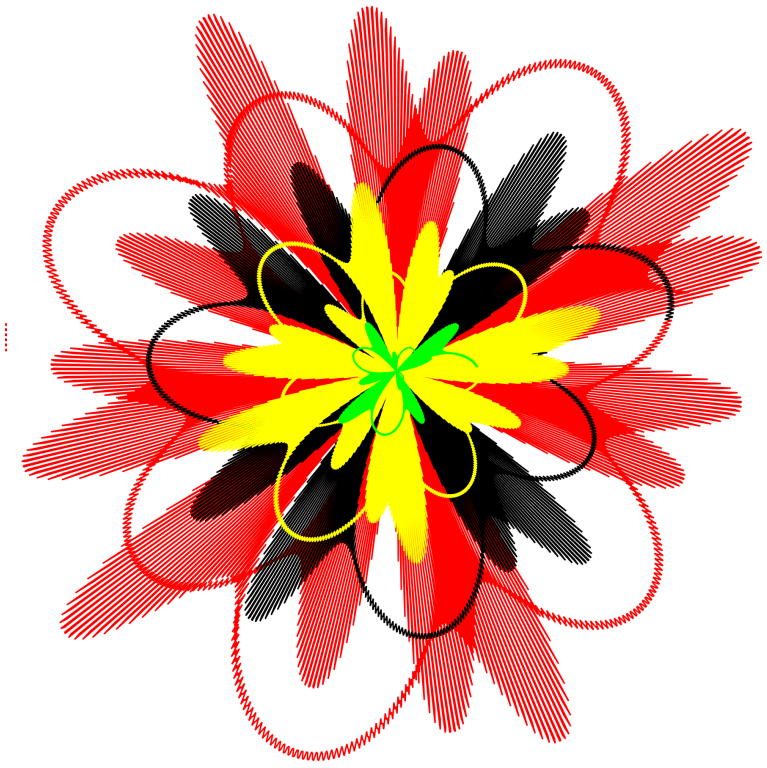




Simulation of Rose Equations

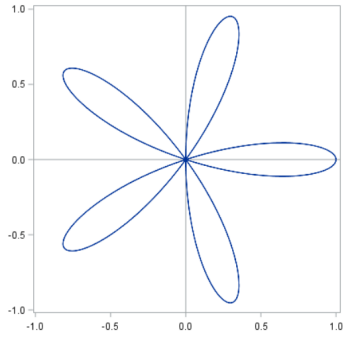
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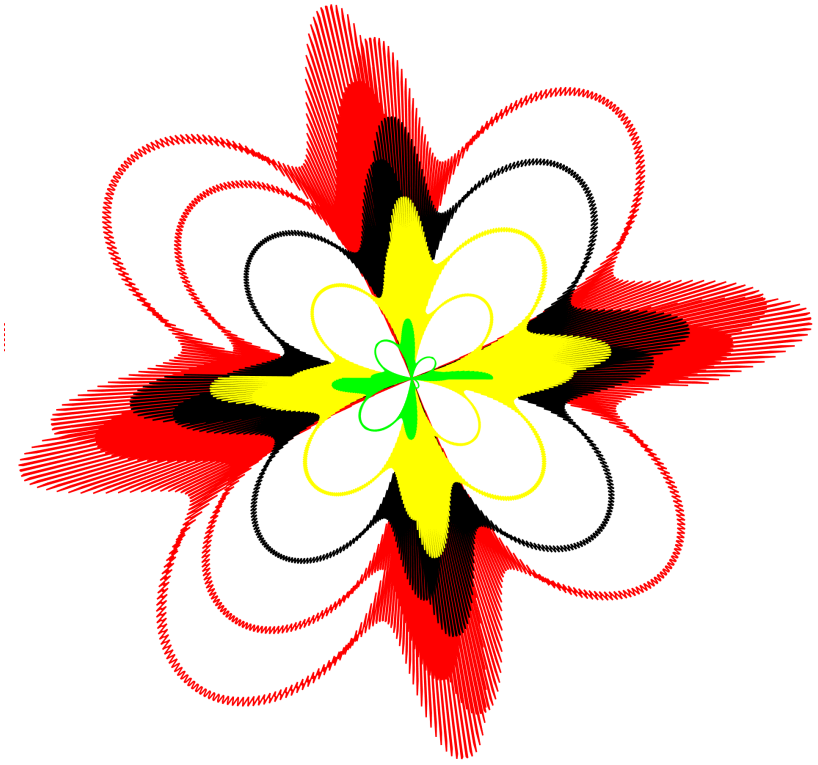




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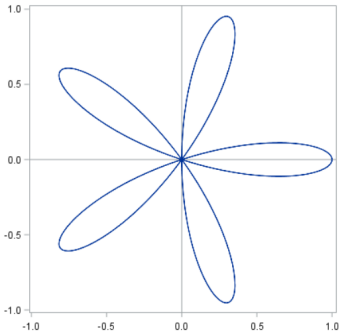
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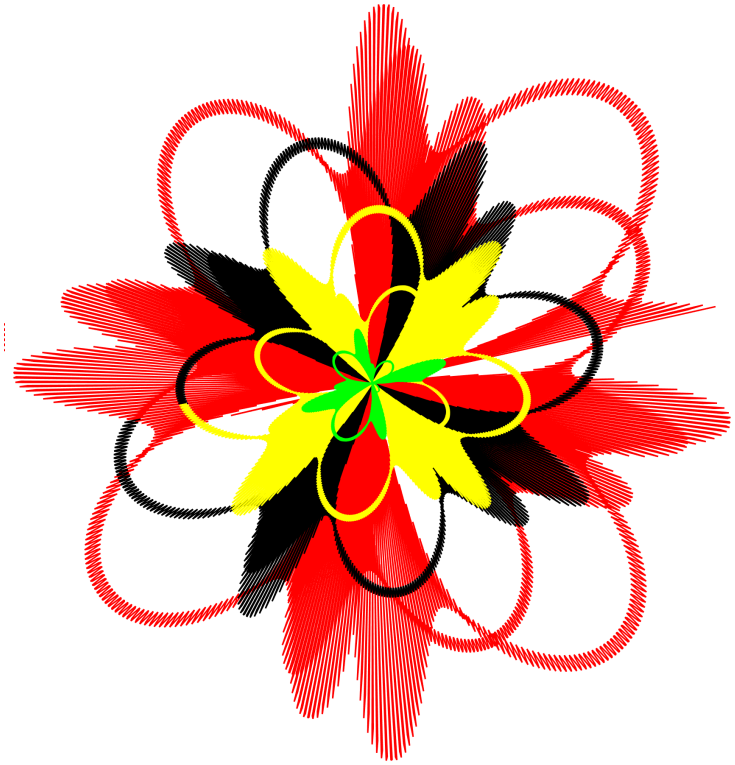




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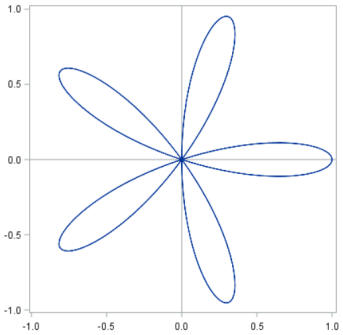
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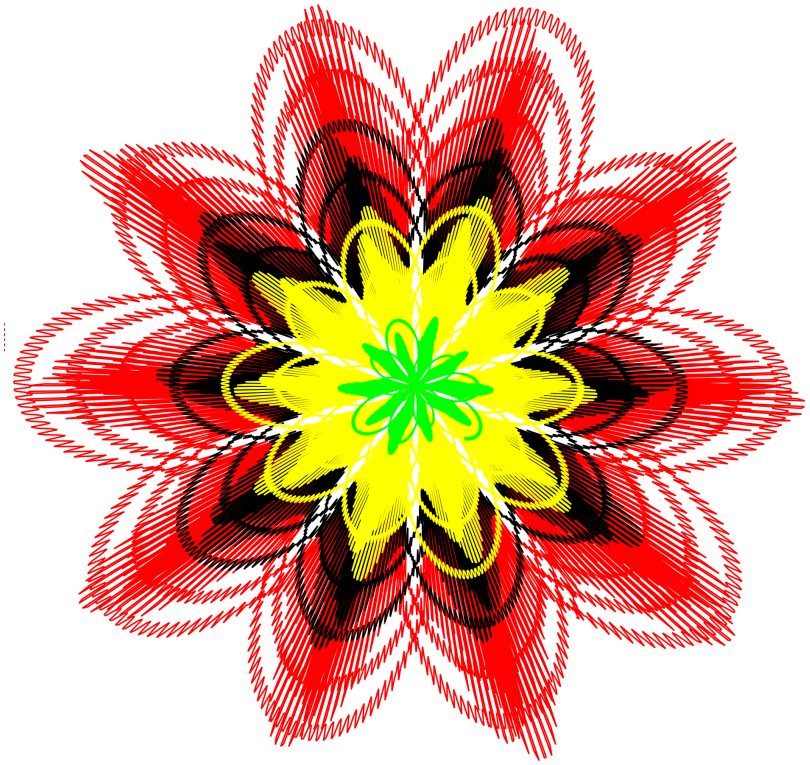




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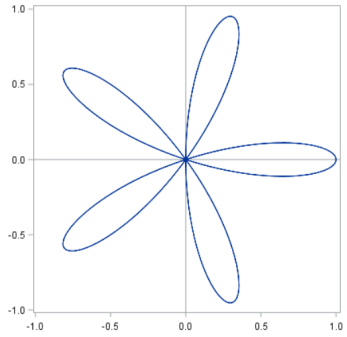
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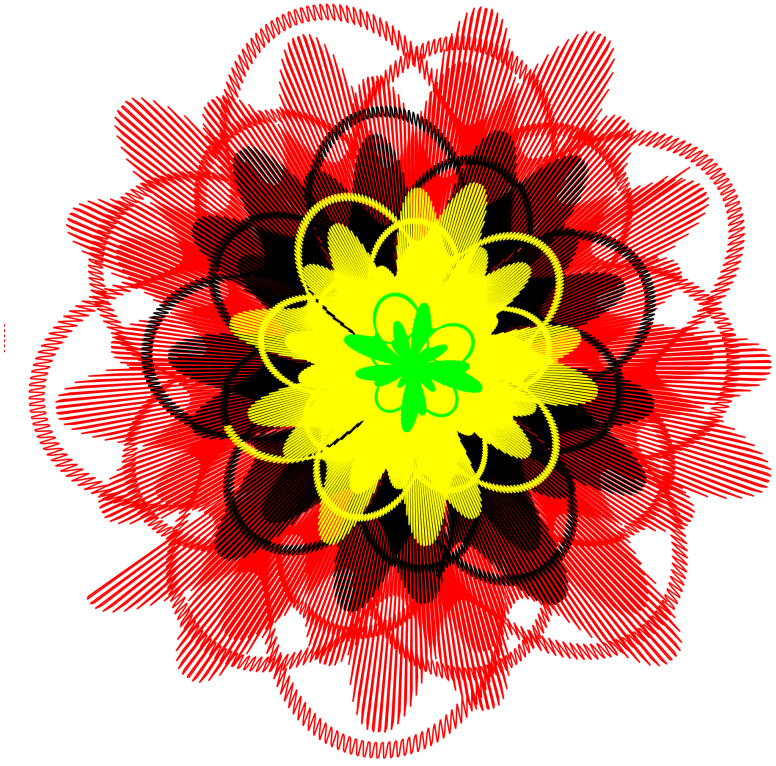




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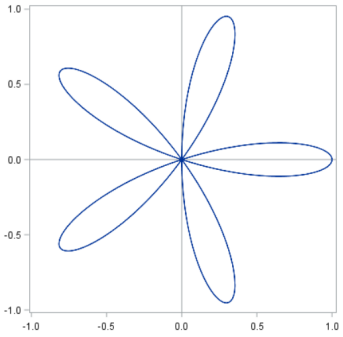
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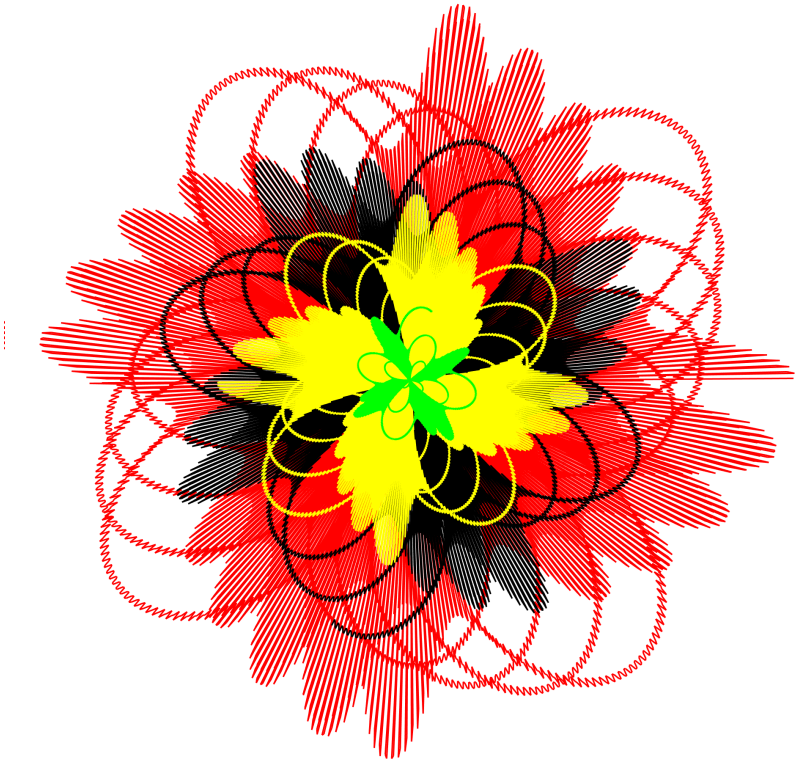




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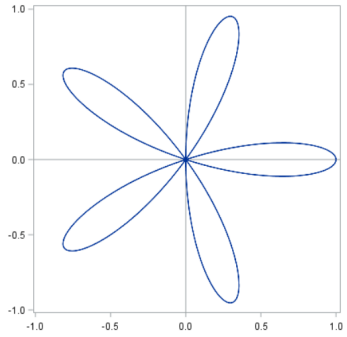
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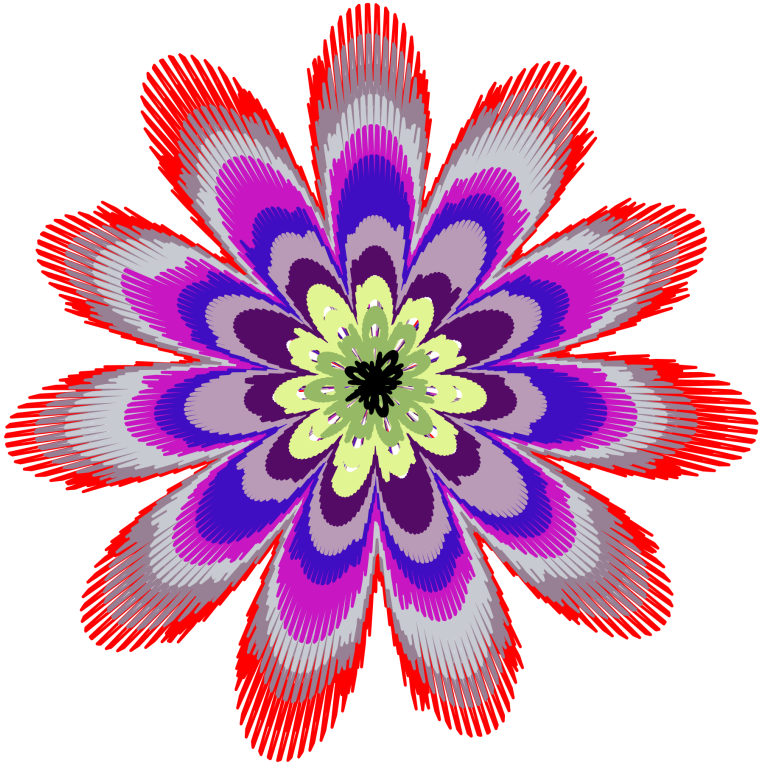




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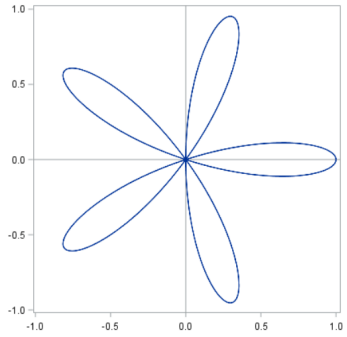
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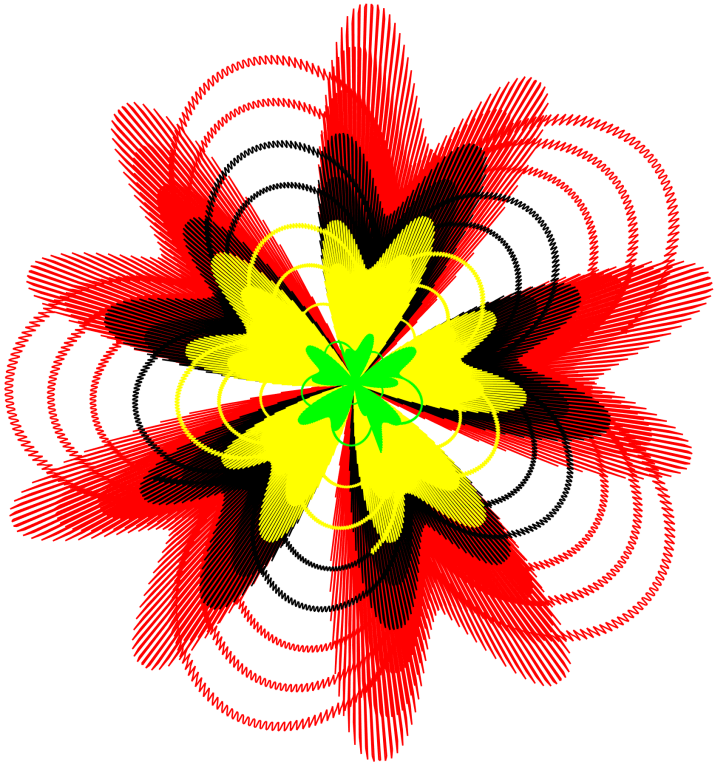




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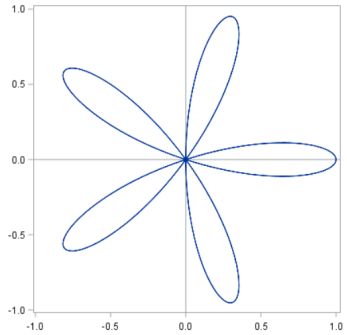


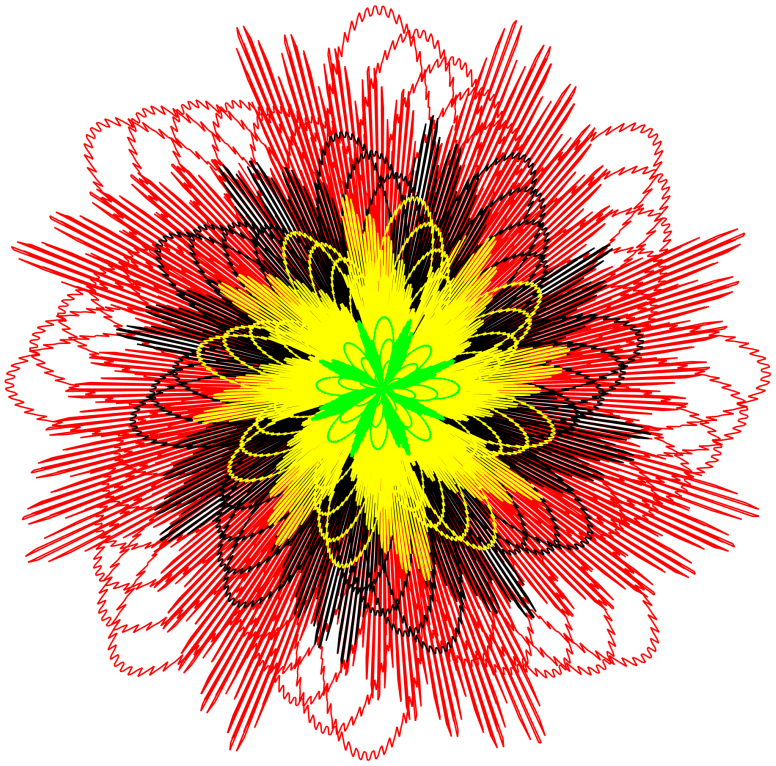


Simulation of Rose Equations using Periodic Sequences

Simulation of Rose Equations

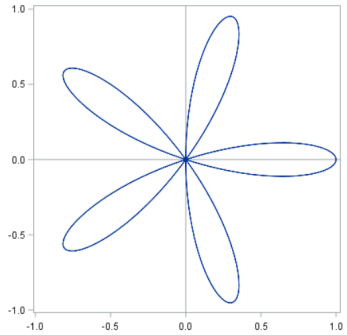
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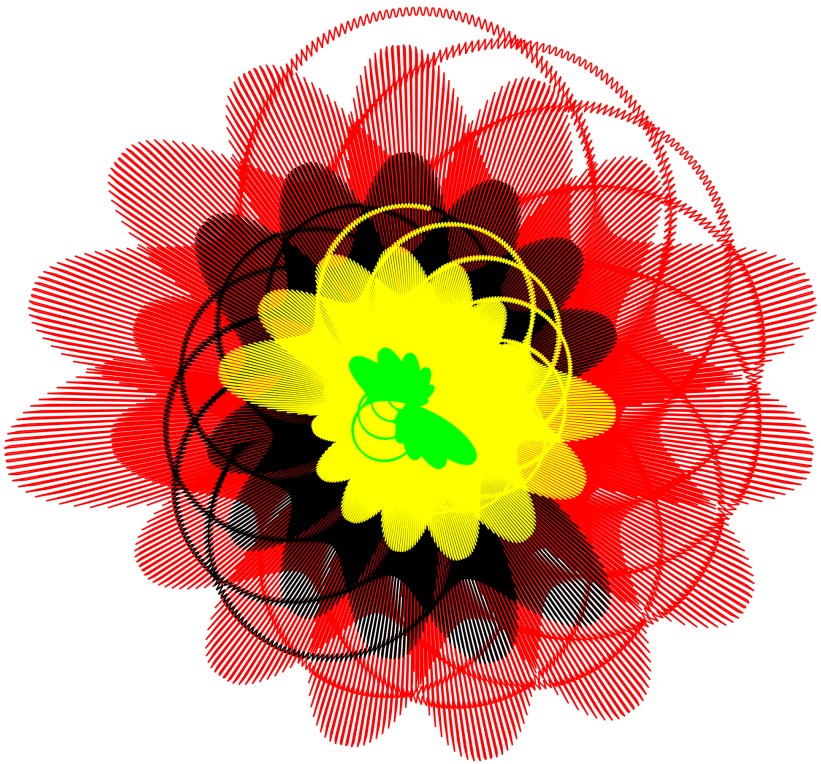




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Concluding Remarks:

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3. Mathematical Arts can have a contribution in Artificial Intelligence and Machine Learning ? How ?
   The Robot can identify the flower as flower even if we as Human we have never seen such kind of flowers!

THANK YOU