Numerical Methods for SDEs and SPDEs

Computational Approach

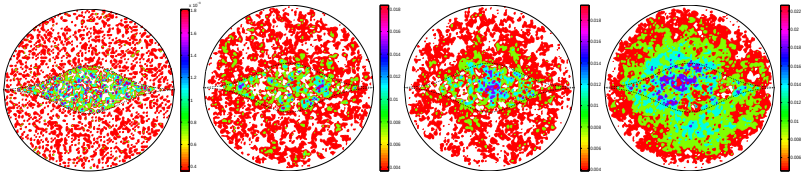
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Numerical Methods for Partial Differential Equations
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Motivation: Diffusion of Bacterial cells as SPDE

Diffusion of Bacterial cells.
Evolution of the solution of a Stochastic Partial differential Equation.

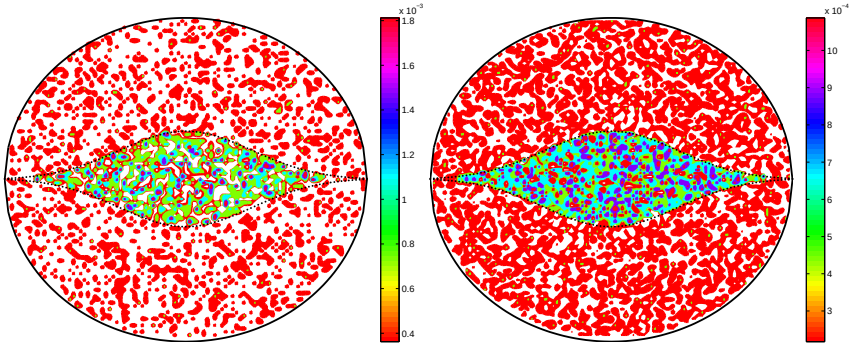


(A) A, t_0 (B) A, t_{100} (C) A, t_{200} (D) A, t_{5000}

(E) B, t_0 (F) B, t_{100} (G) B, t_{200} (H) B, t_{5000}

Abbildung: Under similar initial and boundary conditions, the SPDE solutions are different for different time steps: Tests A and B.

Motivation: Initial vs Mean Solutions of SPDE



(A) $t = 0$ (B) $t = 5000\Delta t, M = 500$

Abbildung: Initial and mean solution at t_{5000} of $M = 500$ realizations, test A.

M. Zahri, Numerical solution of stochastic partial differential systems with additive noise on overlapping subdomains, 7^{th} Inter. Conf. on Approximation Methods and Numerical Modelling in Environment and Natural Resources (MAMERN'17 VII-2017), B. Amaziane et al., pp. 198-201

PART I

Stochastics Differentiation and Integration

- *Random ODEs and Stochastic ODEs*
- *Basics in the theory of Stochastic Processes*
- *Computational Construction of the Brownian motion*
- *Itô formula and Stochastic integrals*
- *Stochastic differential equation*
- *Equivalence SODEs, RODEs and Stratonovich ODEs*

Deterministic vs stochastic Expansion

- The Taylor expansion of a real function $f(x)$ that is infinitely differentiable at a real x_0 is

$$f(x) = \underbrace{\sum_{j\in\mathcal{A}} \frac{(x-x_0)^j}{j!} f^{(j)}(x_0)}_{Approximation\ Term\ P_n(x)} + \underbrace{\sum_{j\in\mathcal{B}(\mathcal{A})} \frac{(x-x_0)^j}{j!} f^{(j)}(x_0)}_{Remainder\ Term} \tag{1}$$

where $\mathcal{A} := \{0, 1, \ldots, p\}$ and $\mathcal{B}(\mathcal{A})$ represents the 'classical' complement of $\mathcal{A}$ in $\mathbb{N}_0$.

- The stochastic Taylor-Expansion of a real valued stochastic process function $f(t, X_t)$ at measurable point (τ, X_τ)

$$f(\tau, X_\tau) = \underbrace{\sum_{\alpha\in\mathcal{A}} I_\alpha[\, f_\alpha(\rho, X_\rho)]}_{Approximation\ Term} + \underbrace{\sum_{\alpha\in\mathcal{B}(\mathcal{A})} I_\alpha[\, f_\alpha(., X.)]_{\rho,\tau},}_{Remainder\ Term} \tag{2}$$

where ρ and τ are Stop-times satisfying ..., X_t is an adapted Itô process, $\mathcal{A}$ is an index set of the main approximation term and $\mathcal{B}(\mathcal{A})$ represents the complement of $\mathcal{A}$.

Starting from the ODE $x'(t) = f(t, x(t))$, we write the deterministic Taylor by using differential operators

$$\begin{aligned} x''(t) &= \frac{d}{dt} f(t, x(t)) = L f(t, x(t)) \\ x'''(t) &= \frac{d}{dt} L f(t, x(t)) = L^2 f(t, x(t)) \\ &\vdots \\ x^{(j)}(t) &= L^{j-1} f(t, x(t)), \quad j = 1, 2, \dots \\ \textit{Set} \qquad L^0 f(t, x(t)) &= f(t, x(t)) \quad \text{and} \quad L^{-1} f(t, x(t)) = x(t) \end{aligned}$$

Hence, the Taylor expansion of $x(t)$ on the neighborhood of t_{n+1} for a given discretization $t = t_0 < t_1 < \cdots < t_N = T$; $h_n = \Delta t_n$ is given by

$$\boxed{x(t_{n+1}) = \sum_{j \in \mathcal{A}} L^{j-1} f(t_n, x(t_n)) \frac{h^j}{j!} + \sum_{j \in \mathcal{B}(\mathcal{A})} L^{j-1} f(t_n, x(t_n)) \frac{h^j}{j!}} \tag{3}$$

where $\mathcal{A} := \{0, 1, \dots, p\}$ and $\mathcal{B}(\mathcal{A})$ represents the 'classical' complement of $\mathcal{A}$ in $\mathbb{N}_0$.

- Random ordinary differential equations (RODEs) are pathwise ODEs that contain a stochastic process in their vector field functions.

$$x'(t) = f(\xi(\omega), x(t)) \Longleftrightarrow x'_\omega(t) = g_\omega(t, x(t)) \quad \text{for} \quad \omega \in \Omega \quad (a.s)$$

- Clearly, once ω is fixed, all classical numerical schemes for ODEs are pathwise applicable for RODEs. (lost of convergence orders!)
- Stochastic ordinary differential equations (SODEs) or (SDEs) are differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is itself a stochastic process.
- Consequently, classical numerical schemes for ODEs are not directly applicable for SDE. (Mathematical transformation are required!)

SO, we need stochastic Differentiation and Integration.

- Goal: Approximation Methods for Stochastic Processes.
- The stochastic processes, solution of Stochastic ODEs, are continuous BUT! Not differentiable!
- How could, we use the stochastic Taylor expansions of solutions with a new concept of differentiability!
- How could we define the differentiability for stochastic processes!
- The intervention of Itô formula is a good start! Especially in integral form.
- The Itô-Taylor expansions have sense by using some appropriate modifications.

We start by setting up some basics on the Theory of Stochastic Calculus.

Sigma-Algebra

Let Ω be a non-empty set and denote by $\mathcal{P}(\Omega)$ the power set. The set $\mathcal{A} \subseteq \mathfrak{P}(\Omega)$ is called σ-Algebra, if it satisfies the following conditions:

(i) $\Omega \in \mathcal{A}$,

(ii) $B \in \mathcal{A} \Rightarrow B^c \in \mathcal{A}$,

(iii) $B_i \in \mathcal{A},\ i = 1, 2, 3, \ldots \Rightarrow \bigcup_i B_i \in \mathcal{A}$.

Examples

(i) If $\Omega \subset \mathbb{N}$ then $\{\phi, \Omega\}$ and $\mathcal{P}(\Omega)$ are sigma-algebras of subsets of Ω.

(ii) If $\Omega = \{1, 2, 3\}$ then $\mathcal{A} = \Big\{1, \{1, 2\}, \Omega, \{2\}, \{1, 3\}, \phi, \{3\}, \{2, 3\}\Big\}$.

(ii) If $\Omega = \mathbb{R}$ then the set $\mathcal{A}$ of all sets of the form $[a, b]$, $(a, b]$, (a, b) and $[a, b)$ is a sigma-algebra, where $a, b \in \mathbb{R}$.

Probability Map

A probability measure P is a map with the following proprieties:

(i) $P : \mathcal{A} \longrightarrow [0, 1]$, (**positivity property**).

(ii) $P(\Omega) = 1$, (**sure event**).

(iii) For any pairwise disjoint $(B_i)_{i \in N} \in \mathcal{A}$, it holds $P\big(\bigcup_{i \in \mathbb{N}} B_i\big) = \sum_{i \in \mathbb{N}} P(B_i)$. (**countably additive** (a.k.a. **sigma-additive**)).

- The properties above are usually referred to the **Kolmogorov Axioms**.
- A probability map is also called a **probability function**.
- The probability of an **impossible event** is always zero 0 :The probability of getting a number greater than 6, when a die is thrown once, is an **impossible event**, as the highest number in a die is 6.

- The Tuple $(\Omega, \mathcal{A})$ is called **measurable space**: The space of measurable events
- The Triple $(\Omega, \mathcal{A}, P)$ is called **probability space**: Non-deterministic space
- The real valued map $X : \Omega \to \mathbb{R}$ is called **random variable**, if X is $\mathcal{A}-$measurable: For all $t \in \mathbb{R}$, it holds

$$\left\{X \leqslant t\right\} := X^{-1}(]-\infty, t]) = \left\{\omega \in \Omega : X(\omega) \in]-\infty, t]\right\} \in \mathcal{A},$$

  In other words: The History of X till t are measurable events ie. are in $\mathcal{A}$.
- If $d \in \mathbb{N}$ and $X^1, \ldots, X^d$ are random variables defined on the same P.S., the $X = (X^1, \ldots, X^d)$ will be called **random vector**.

A family $(\mathcal{F}_t)_{t\in\mathbb{T}}$ ($\mathbb{T} = \mathbb{N}$ or $[0,\infty)$) of σ-Algebras is called **filtration** if the following properties are satisfied:

(*i*) for all $t \in \mathbb{T}$, $\mathcal{F}_t \subset \mathcal{A}$, (**measurability property**)

(*ii*) for all $s,t \in \mathbb{T} = \mathbb{N} : s \leqslant t \ \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t$, (**increasing measurable information**)

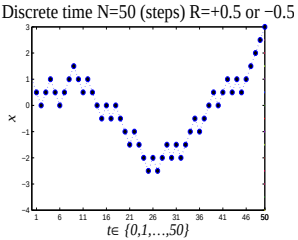
It is important to know that:

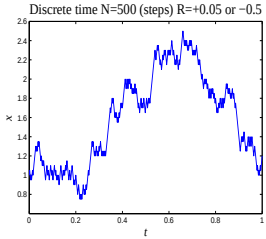
- A filtration $\mathcal{F}$ is right continuous if $\forall\, t \in \mathbb{T},\ \mathcal{F}_t = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$,
- $\mathcal{F}_0$ contains the zero-set A, if $\{A : A \in \mathcal{A}, P(A) = 0\} \subseteq \mathcal{F}_0$.

Consider a probability space $(\Omega, \mathcal{A}, P)$ and an index set $\mathbb{T} \subseteq \mathbb{R}^+$.

- The map $X : \mathbb{T} \times \Omega \to \mathbb{R}$ is called **stochastic process** if $X_t : \Omega \to \mathbb{R}$ is a random variable for each $t \in \mathbb{T}$. $(X_t(\omega) \equiv X(t, \omega))$.
- If $\mathbb{T} = \mathbb{N}_0$ then X_t is called a **discrete time** stochastic process.
- If $\mathbb{T} = \mathbb{R}^+$ then X_t is called a **continuous time** stochastic process.
- A d**-dimensional stochastic process** is a family of d-dimensional random vectors $(X_t)_{t\in\mathbb{T}}$. $(d \in \mathbb{N})$.
- The process $(X_t)_{t\in\mathbb{T}}$ is measurable, if the process $\mathbb{T} \times \Omega \to \mathbb{R}^d$, $(t, \omega) \mapsto X_t(\omega)$ is $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}) - \mathcal{B}(\mathbb{R}^d)$ measurable.
- The process $(X_t)_{t\in\mathbb{T}}$ is called **progressive measurable** with respect to the filtration $\mathcal{F}$, if for any $t \in \mathbb{T}$ the process $[0, t] \times \Omega \to \mathbb{R}^d$, $(s, \omega) \mapsto X_s(\omega)$ is $(\mathfrak{B}([0, t]) \otimes \mathfrak{F}_t) - \mathfrak{B}(\mathbb{R}^d)$ measurable.

For any fixed $\omega \in \Omega$ the function $\mathbb{T} = \mathbb{N}_0 \to \mathbb{R}^d, \quad t \mapsto X_t(\omega)$ is called a **Realization** or **Trajectory** of the process $(X_t)_{t\in\mathbb{T}}$

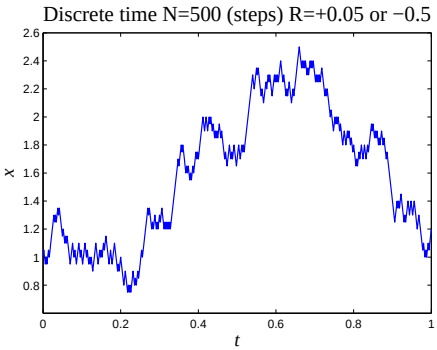


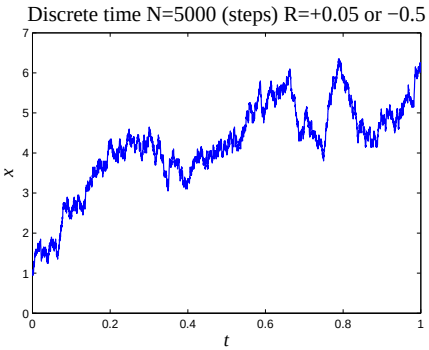


- Start point $X_0 = 1$ (Not random)
- The process X_t is generated by using a Bernoulli Distribution B(0.5).
- $X_{t+1} = X_t + R(\sim B(0.5))$.
- $R = +0.5$ or -0.5 up-down

The figure above shows two different realizations of the time discrete stochastic process X_t. Passage from discrete to continuous case!

For any fixed $\omega \in \Omega$ a continuous **Realization** of the process $(X_t)_{t\in[0,1]}$ is constructed by generating random numbers for a discretization of the continuous time-interval $[0, 1]$.





- For the left-figure $\Delta t = 1/500$ and for the right-figure $\Delta t = 1/5000$.
- For $t \in (t_0 + j\Delta t, t_0 + (j + 1)\Delta t)$ linear interpolation is applied.
- Continuous means in the computational sense $\Delta t << 1$.

Let $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$ be a filtration and a reel process $X = (X_t)_{t\in\mathbb{T}}$ such that
(i) X is $\mathcal{F}$ adapted (i.e. $\forall t \in \mathbb{T} : X_t$ is $\mathcal{F}_t - measurable$), and
(ii) for all $t \in \mathbb{T} : E(|X_t|) < \infty$.

- X is called **martingale** with respect to the filtration $\mathcal{F}$, if

$$E(X_t|\mathcal{F}_s) = X_s \qquad (a.s)$$

- A map $\rho : \Omega \to \mathbb{T} \cup \{\infty\}$ is called **stop time** with respect to $(\mathcal{F}_t)_{t\in\mathbb{T}}$, if

$$\forall\, t \in \mathbb{T} : \{\rho \leqslant t\} \in \mathcal{F}_t.$$

A one-dimensional **Brownian motion** (also called **standard Wiener process**) is a real-valued stochastic process $\{W_t\}_{t\geqslant 0}$, defined on a probability space $(\Omega, \mathcal{A}, P)$ with filtration $\mathcal{F}$, indexed by nonnegative real numbers t with the following properties:

1. $W_0 = 0$.
2. With probability 1, the function $t \longmapsto W_t$ is continuous in t.
3. The process $\{W_t\}_{t>0}$ has stationary, independent increments.
4. The increment $W_t - W_s$ is normally distributed with mean zero and variance $t - s$ i.e

$$W_t - W_s \sim \sqrt{t-s}\mathcal{N}(0,1), \quad \text{for all} \quad t > s.$$

For $t \in \mathbb{T}$, $m \in \mathbb{N}$, a **m-dimensional Brownian motion** is given as:

$$W_t = \left(W_t^1, \ldots, W_t^m\right)^T,$$

where

- For all $i = 1, \ldots, m$, $(W_t^i)_{t \in T}$ are independent one dimensional Brownian motions.
- For $0 \leqslant t_0 < \ldots < t_{n-1} < t_n$ the increments $(W_{t_1} - W_{t_0}), \ldots, (W_{t_n} - W_{t_{n-1}})$ are independent identicaly distributed (i.i.d)

The Brownian motion was first discovered by the Scottish botanist Robert Brown in 1827. The notion that the increments of the Brownian motion are normally distributed is the source of immense scientific results, either in stochastic analysis or in the interpretation of physical, biological, econometric models. We refer the reader to [2].

Construction of normally distributed numbers

- The Box-Müller method takes independent standard uniform random variables U_1 and U_2 and produces independent standard normals X_1 and X_2 using the formulas:

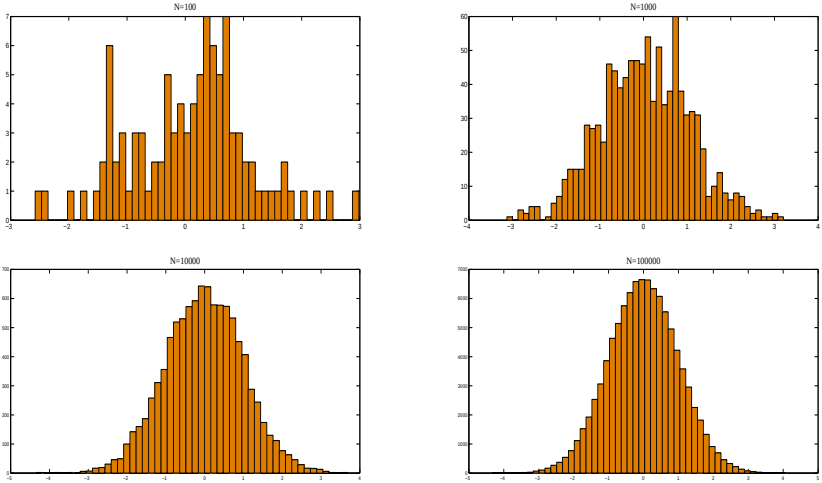
$$\theta = 2\pi U_1, \quad R = \sqrt{-2\ln(U_2)}, \quad X_1 = R\cos(\theta), \quad X_2 = R\sin(\theta). \tag{4}$$

- In other words from two random numbers $u_1, u_2 \in (0, 1]$ (generated by a uniform distribution), we produce two independent standard normally distributed numbers n_1 and n_2, namely:

$$n_1 = \sqrt{-2\ln(u_1)}\cos(2\pi u_2), \qquad n_2 = \sqrt{-2\ln(u_1)}\sin(2\pi u_2). \tag{5}$$

The Box-Müller transform was suggested by George Edward Pelham Box and Mervin Edgar Müller (1958)

Normality test: For $N = 100$, $N = 1000$,$N = 10000$ and $N = 100000$



The Gaussian behavior is clearly justified for large N

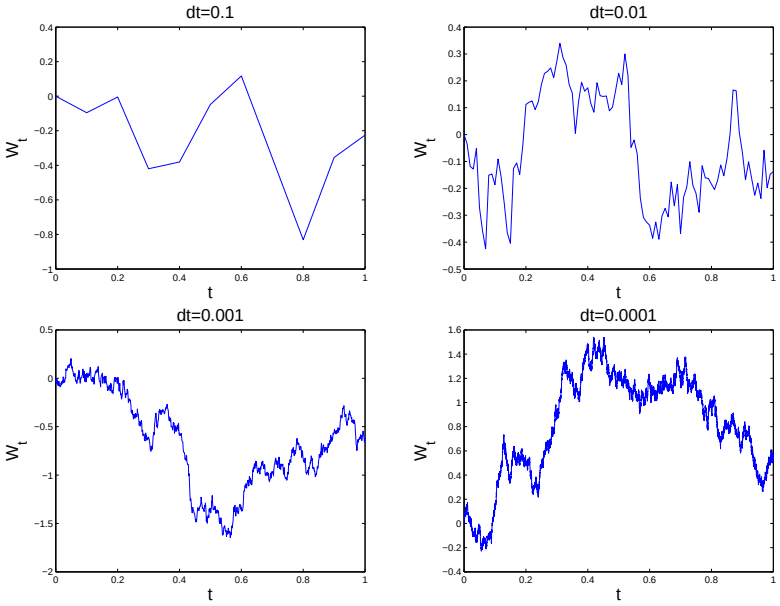
The Brownian motion as a time dependent stochastic process:
Let $0 = t_0 < t_1 < \ldots < t_N = T$ be an equidistant discretization of the time Interval $[0, T]$, i.e. $t_k = k\Delta$ with $\Delta = \frac{T}{N}$. Since the BM increments are $i.i.d$ and normally distributed. we have

$$\boxed{\begin{aligned} W_{t_0} &= W_0 = 0 \\ W_{t_{k+1}} &= W_{t_k} + R\sqrt{\Delta t} \quad \text{where} \quad R \sim \mathcal{N}(0,1). \end{aligned}}$$

In order to simulate the path (a realization) of a Brownian motion, we note that

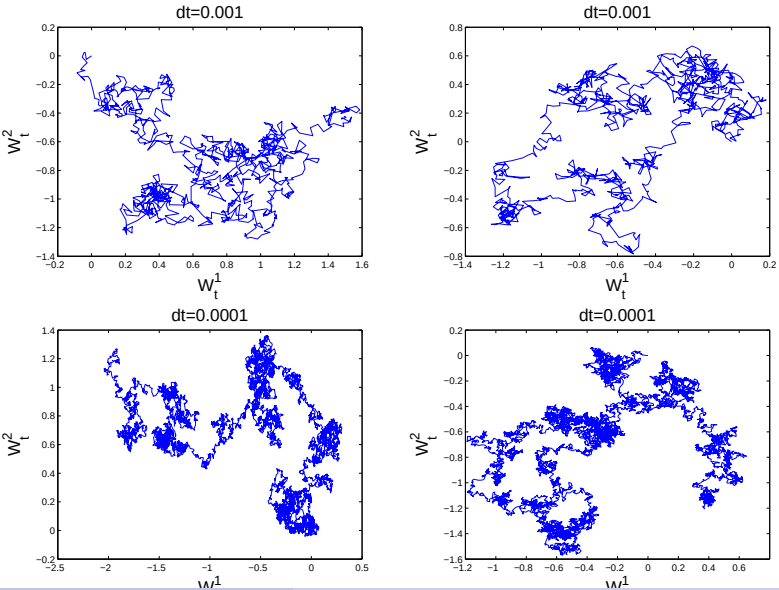
- the values W_{t_k} $\forall k = 0, 1, \ldots, N$ are per recursion obtained,
- for all $t \in]t_k, t_{k+1}[$, W_t are linearly interpolated.

Using different time steps, the following figures shows for sample of the BM.

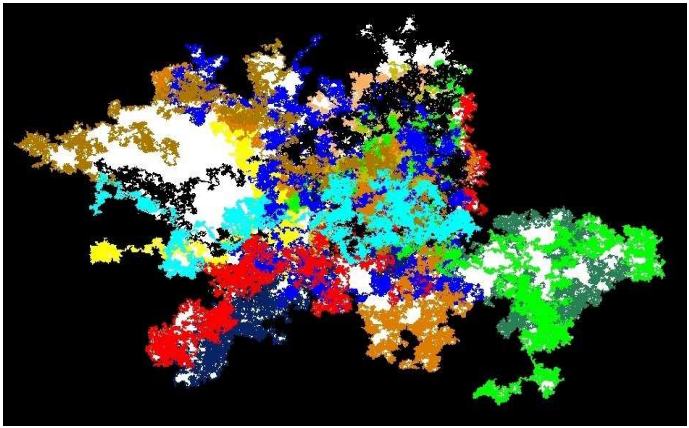


Two-dimensional noise-like process:

Using different time steps, the following figures shows for sample of the planar BM:

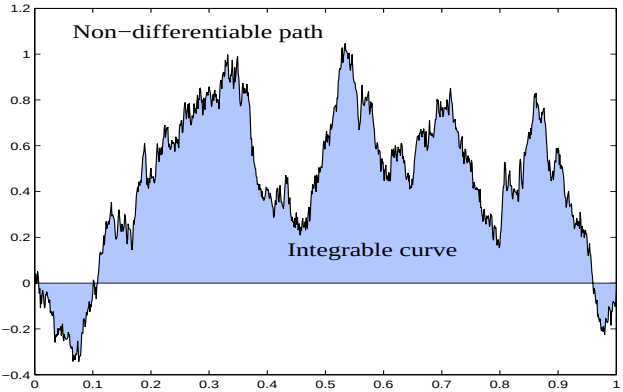


Overlapping two-dimensional noise-like processes:



The two dimensional plot of several planar Brownian motions by using different colors.

The integration of the solutions in time is achieved by using the Itô formula in integral form instead of the nondifferentiability (mathematical problem).



The figure above shows a nowhere differentiable stochastic process and a bounded area under the curve.

Therefore integration instead of differentiation.

Let $(\Omega, \mathcal{A}, P)$ be a probability space with Filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$. For $T > 0$, the set Λ_T is the set of $\mathcal{F}$-adapted stochastic process such that $E\left(\int_0^T Z_s^2 ds\right) < \infty$. The **Itô-Integral** with respect to BM is given as:

$$I[Z]_{0,t} := \int_0^t Z_s dW_s = \lim_{N\to\infty} I^{(N)}[Z]_{0,t}$$

(similar to the deterministic case) where $Z \in \Lambda_T$ and $I^{(N)}[Z]_{0,t} = \sum_{k=1}^{N} Z_{t_{k-1}^{N,l}}(W_{t_k^{N,l}} - W_{t_{k-1}^{N,l}})$, for $\tau_N^l = \{t_k^{N,l} : k = 0, \ldots, N \text{ und } l \in \mathbb{N}\}$ a discretization sequence of $[0, t]$.

- The limit $\lim_{N\to\infty} E(I[Z]_{0,t} - I^{(N)}[Z]_{0,t})^2 = 0$ exists in $L^2(P)$ and it is unique for all $t \in [0, T]$. (See [6]).

Consider a one dimensional Brownian motion and a non-necessary uniform time discretization $t_k = k\frac{T-t_0}{2^n}$ of the interval $[t_0, T]$. Then we have,

$$1). \quad \lim_{n\to\infty} \sum_{k=0}^{2^n-1} (\Delta t_k)^2 = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \Delta t_k \Delta W_{t_k} = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \Delta W_{t_k} \Delta t_k = 0.$$

$$2). \quad \lim_{n\to\infty} \sum_{k=0}^{2^n-1} (\Delta W_{t_k})^2 = \int_{t_0}^{T} ds = (T - t_0). \qquad \text{(Convergence in } L^2\text{)}$$

where $\Delta t_k = t_{k+1} - t_k$ and $\Delta W_{t_k} = W_{t_{k+1}} - W_{t_k}$.
Per passage to the continuous case, this means

$$\boxed{dtdt = dtdW_t = dW_t dt = 0},$$

$$\boxed{dW_t dW_t = dt.}$$

Using the definition of the Itô, the following are one dimensional Itô integral (OSI):

$$\int_0^t dW_s = \lim_n \sum_{i=0}^{n} \left(W_{t_{i+1}} - W_{t_i}\right) = W_t - W_0$$

$$\int_0^t W_s dW_s = \lim_n \sum_{i=0}^{n} W_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)$$

$$= \lim_n \sum_{i=0}^{n} W_{t_i} \Delta W_{t_i}$$

$$(W_t)^2 - 2W_{t_i}\Delta W_{t_i} = \sum_{i=0}^{n} \left(\Delta W_{t_i}\right)^2$$

$$= \frac{1}{2}(W_t)^2 - \frac{1}{2} \underbrace{\lim_n \sum_{i=0}^{n} \left(\Delta W_{t_i}\right)^2}_{\to t \quad (\text{in } L^2)} \qquad \text{later !!!}$$

$$= \frac{1}{2}(W_t)^2 - \frac{1}{2}t$$

Clearly the behavior of the second integral is different from the Riemann-Stieltjes

- **[Linearity]** Consider $(Z_t^{(1)})_{t\in\mathbb{T}}, (Z_t^{(2)})_{t\in\mathbb{T}} \in \Lambda_T$ and $K_1, K_2 \in \mathbb{R}$. For $I[Z^{(1)}]_{0,t} = \int_0^t Z_s^{(1)} dW_s$ and $I[Z^{(2)}]_{0,t} = \int_0^t Z_s^{(2)} dW_s$, it holds

$$\boxed{I[K_1 Z^{(1)} + K_2 Z^{(2)}]_{0,t} = K_1 I[Z^{(1)}]_{0,t} + K_2 I[Z^{(2)}]_{0,t}}$$

- **[Martingale]**

$$\boxed{E(I[Z]_{0,t}|\mathcal{F}_s) = I[Z]_{0,s}, \quad \forall s,t, \quad 0 < s < t.}$$

- **[Itô Isometry]**

$$\boxed{E(I[Z]_{0,t})^2 = E\Big(\int_0^t Z_u^2 du\Big), \quad \forall t, \quad 0 \leqslant t \leqslant T.}$$

- **[Continuity of I_t]** The process $I[Z]_{0,t}$ is continuous with respect to t: It exists a continuous process h_t such that

$$\boxed{P\left(h_t = I[Z]_{0,t}\right) = 1 \quad \forall t, \quad 0 \leqslant t \leqslant T}$$

For $m \in \mathbb{N}$ and $F = \{0, 1, \ldots, m\}$.For $\alpha = (j_1, \ldots, j_l)$ where $j_i \in F$ and $1 \leqslant i \leqslant l$. We denote the **size** and the **number of zero components**of α

$$\begin{aligned} \ell(\alpha) &:= \ell_\alpha \qquad number\ of\ components\ of\ \alpha \\ n(\alpha) &:= n_\alpha \qquad number\ of\ zero\ components\ of\ \alpha \end{aligned}$$

The set of all multi-indices with respect to F is represented by

$$\mathcal{M} = \bigcup_{l=1}^{\infty} F^l \cup \{v\}, \tag{6}$$

where v refers to the empty multi-index with size zero. The following example gives more sense for the definition above:

EXAMPLE

For $\alpha = (1, 0, 2)$ it holds $\ell(\alpha) = 3 \quad$ and $\quad n(\alpha) = 1$.
For $\alpha = (1, 0, 0, 2, 3, 1, 0, 0)$ it holds $(\alpha) = 8 \quad$ and $\quad n(\alpha) = 4$.

- ($-$ operator). For $\alpha \in \mathcal{M}$ with $\alpha = (j_1, j_2, \ldots, j_l)$. For $l \geqslant 1$, we define $\alpha-$ and $-\alpha$ as follows:

$$\begin{aligned} \alpha- &:= (j_1, j_2, \ldots, j_{l-1}) \quad \text{drop last component.} \\ -\alpha &:= (j_2, \ldots, j_l) \quad \text{drop last component.} \end{aligned}$$

  If $\ell(\alpha) = l > 1$ then, it implies $l(-\alpha) = l(\alpha-) = l - 1$.
  If $\ell(\alpha) = l = 1$ then, it implies $-\alpha = \alpha- = v$ and $l(-\alpha) = l(\alpha-) = 0$.

- ($\star$ operator). Let us consider $\alpha = (j_1, j_2, \ldots, j_l), \beta = (i_1, i_2, \ldots, i_k) \in \mathcal{M}$. The operator $\star$ is defined as:

$$\begin{aligned} \alpha * \beta &:= (j_1, j_2, \ldots, j_l, i_1, i_2, \ldots, i_k) \quad \text{and} \\ \beta * \alpha &:= (i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_l). \end{aligned}$$

Define the set H as a set of stochastic processes $(f_t)_{t\geqslant 0}$, which are progressively adapted to the associated filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$), right continuous and the left limit exists. Moreover, define the sets $H_v, H_{(0)}, H_{(1)}$ as follows

1. $H_v := \{f \in H : \forall\, t \geqslant t_0 \quad |f(t,w)| < \infty \quad \text{a.s.}\}$,
2. $H_{(0)} := \left\{f \in H : \forall\, t \geqslant t_0 \quad \int_{t_0}^{t} |f(s,w)|ds < \infty \quad \text{a.s.}\right\}$,
3. $H_{(1)} := \left\{f \in H : \forall\, t \geqslant t_0 \quad \int_{t_0}^{t} |f(s,w)|^2 ds < \infty \quad \text{a.s.}\right\}$.

For $j \in F\setminus\{0\}$ one sets $H_{(j)} = H_{(1)}$. Here $H_{(\alpha)}$ is defined per recursion as $H_{(\alpha)} := \{f \in H : \quad I_{(\alpha-)}[f(.)]_{t_0,.} \in H_{(j_l)}\}$, for $j_l \in F$ and $l \geqslant 2$.

Useful Notation for multiple Stochastic Integrals

Let us consider $\alpha = (j_1, j_2, \ldots, j_l)$ a multi-index and $(W_t)_{t \geqslant 0}$ an m-dimensional Brownian motion. For $f \in H_{(\alpha)}$, multiple Itô-Integrals are defined per recursion as follows:

$$I_\alpha[f(.)]_{t_0,t} := \begin{cases} f(t), & \text{if } l = 0 \\ \displaystyle\int_{t_0}^{t} I_{\alpha-}[f(.)]_{t_0,s} ds, & \text{if } l \geqslant 1 \text{ and } j_l = 0 \\ \displaystyle\int_{t_0}^{t} I_{\alpha-}[f(.)]_{t_0,s} dW_s^{j_l}, & \text{if } l \geqslant 1 \text{ and } j_l \geqslant 1. \end{cases}$$

Example

$$\begin{aligned} W_t^0 &= t \quad I_{\alpha,t} = I_\alpha[1]_{0,t} \\ I_{(1,2)}[f(.)]_{t_0,t} &= \int_{t_0}^{t} \int_{t_0}^{s} f(z) dW_z^1 dW_s^2, \\ I_{(1,2,0)}[f(.)]_{t_0,t} &= \int_{t_0}^{t} I_{(1,2)}[f(.)]_{0,s} ds = \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{s_1} f(s_2) dW_{s_2}^1 dW_{s_1}^2 ds. \end{aligned}$$

Recurrence relationship for I_α, $\alpha = (j_1, j_2, \ldots, j_n) \in \mathcal{M}$

Theorem

Let us consider $l \in \mathbb{N}$ and $\alpha = (j_1, ..., j_l) \in \mathcal{M}$. For $t \geqslant 0$, we have

$$I_{(j),t} I_{(\alpha),t} = \sum_{i=0}^{l} I_{(\alpha-[l-i])*(j,j_{i+1},...,j_l),t} + \sum_{i=1}^{l} B_{jj_i} I_{(\alpha-[l-i+1])*(0,j_{i+1},...,j_l),t},$$

where $B_{jj_i} = \delta_{j,j_i}(1 - \delta_{0,j})$.

Recurrence relationship for I_α, $\alpha = (j, j, \ldots, j) \in \mathcal{M}$

Corollary

Consider $l, j \in \mathbb{N}$ and $\alpha = (j, j, \ldots, j)$ with $l(\alpha) = l$. It holds:

$$I_{\alpha,t} = \begin{cases} \frac{t^l}{l!} & \text{for } j = 0, \\ \frac{1}{l}(W_t^j I_{\alpha-,t} - t I_{\alpha-[2],t}) & \text{for } j \geqslant 1. \end{cases}$$

Lemma

We have the following values of the multiple stochastic integrals for the special case $\alpha = (j, j, \ldots, j) \in \mathcal{M}$:

$$I_{(j,j),t} = \frac{1}{2}(I^2_{(j),t} - t) = \frac{1}{2}\Big((W^j_t)^2 - t\Big). \tag{7}$$

$$I_{(j,j,j),t} = \frac{1}{3}\Big(I_{(j),t}\frac{1}{2}(I^2_{(j),t} - t) - tI_{(j),t}\Big) = \frac{1}{3!}\Big(I^3_{(j),t} - 3tI_{(j),t}\Big), \tag{8}$$

$$I_{(j,j,j,j),t} = \frac{1}{4!}\Big(I^4_{(j),t} - 6tI^2_{(j),t} + 3t^2\Big), \tag{9}$$

$$I_{(j,j,j,j,j),t} = \frac{1}{5!}\Big(I^5_{(j),t} - 10tI^3_{(j),t} + 15t^2I_{(j),t}\Big), \tag{10}$$

$$I_{(j,j,j,j,j,j),t} = \frac{1}{6!}\Big(I^6_{(j),t} - 15tI^4_{(j),t} + 45t^2I^2_{(j),t} - 15t^3\Big), \tag{11}$$

$$I_{(j,j,j,j,j,j,j),t} = \frac{1}{7!}\Big(I^7_{(j),t} - 21tI^5_{(j),t} + 105t^2I^3_{(j),t} - 105t^3I_{(j),t}\Big). \tag{12}$$

If $W_t^{j_1}$ and $W_t^{j_2}$ are two independent Brownian motions the MSI

$$I_{(j_1,j_2)t_0,t} = \int_{t_0}^{t}\int_{t_0}^{s_1} dW_s^{j_1} dW_{s_1}^{j_2},$$

can not be evaluated exactly.
Some proposed approximation could be found in literature

- One can use of the periodic concept of Brownian Bridge and the Fourier series gives another approximation method.
- direct evaluation of the variation given by:

$$\hat{I}_{(j_1,j_2)t_0,t} = \sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\int_{t_0}^{t_i} dW_s^{j_1} dW_{s_1}^{j_2} = \sum_{i=0}^{m-1}(W_{t_i}^{j_1} - W_{t_0}^{j_1})(W_{t_{i+1}}^{j_2} - W_{t_i}^{j_2}) \qquad (13)$$

- other methods... Literature

Therefore, $\alpha = (j_1, j_2, \ldots, j_n) \in \mathcal{M}$ for at least $j_i \neq j_k$ components of α the MSI I_α, has no exact analytical expansion, Thus approximation methods are needed.

Let $(W_t)_{t\in\mathbb{T}}$ be an m-dimensional Brownian motion defined on a $(\Omega, \mathcal{A})^m$ with right continuous augmented filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$. The process $(X_t^1 \ldots, X_t^d)$ is called **Itô Process**, iff it has the following form

$$X_t^i = X_{t_0}^i + \underbrace{\int_{t_0}^t a_s^i ds}_{Riemann-integral} + \underbrace{\sum_{j=1}^m \int_{t_0}^t b_s^{i,j} dW_s^j}_{Itô-integral}, \quad (SDE)$$

where for all $i = 1, \ldots, d$, and $j = 1, \ldots, m$; the **drift vector** $(a_t^i)_{t\in\mathbb{T}}$ and the **diffusion matrix** $(b_t^{i,j})_{t\in\mathbb{T}}$ are $\mathcal{F}_t$ adapted and satisfy $\int_{t_0}^T a_s^i ds < \infty$ and $\int_{t_0}^T (b_s^{i,j})^2 ds < \infty$ a.s.

- The (SDE) refers to the **stochastic differential equation** in integral form and in differential form is given by

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

Let us consider the functional $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}$ with continuous partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x^i}$ and $\frac{\partial^2 f}{\partial x^i \partial x^j}$ for $i = 1, \ldots, d$. Moreover, consider a d-dimensional Itô-Process $(X_t)_{t\in\mathbb{T}}$, then we have,

$$\begin{aligned} f(t, X_t^1, \ldots, X_t^d) = f(t_0, X_{t_0}^1, \ldots, X_{t_0}^d) &+ \int_{t_0}^t \frac{\partial f}{\partial s}(s, X_s^1, \ldots, X_s^d)ds \\ &+ \sum_{i=1}^d \int_{t_0}^t \frac{\partial f}{\partial x^i}(s, X_s^1, \ldots, X_s^d)dX_s^i \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_{t_0}^t \frac{\partial^2 f}{\partial x^i x^j}(s, X_s^1, \ldots, X_s^d)d < X^i, X^j >_s, \end{aligned}$$

where $d < X^i, X^j >_s = \sum_{k=1}^m b^{i,k}(s, X_s)b^{j,k}(s, X_s)ds$

The one dimensional case $d = m = 1$ of the Itô-Formula is given as

$$f(t, X_t) = f(t_0, X_{t_0}) + \int_{t_0}^{t} \left\{ \frac{\partial f}{\partial s}(s, X_s) + a_s \frac{\partial f}{\partial x}(s, X_s) + \frac{1}{2} b_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s) \right\} ds + \int_{t_0}^{t} b_s \frac{\partial f}{\partial x}(s, X_s) dW_s.$$

Proof idea: By using the Taylor expansion of order two: (all convergence in L^2)

$$\begin{aligned} f(t, X_t) &= f(t_0, X_{t_0}) + \sum_{k=0}^{2^n-1} \Delta f(t_k, X_{t_k}) \\ &= f(t_0, X_{t_0}) + \underbrace{\sum_{k=0}^{2^n-1} \frac{\partial f}{\partial t} \Delta t_k}_{\to \int_{t_0}^{t} \frac{\partial f}{\partial t} ds} + \underbrace{\sum_{k=0}^{2^n-1} \frac{\partial f}{\partial x} \Delta X_{t_k}}_{\to \int_{t_0}^{t} \frac{\partial f}{\partial x} a_s ds + \int_{t_0}^{t} \frac{\partial f}{\partial x} b_s dW_s} + \frac{1}{2} \underbrace{\sum_{k=0}^{2^n-1} \frac{\partial^2 f}{\partial x^2} (\Delta X_{t_k})^2}_{\to \int_{t_0}^{t} \frac{\partial^2 f}{\partial x^2} b_s^2 ds} \\ &\quad + \underbrace{\sum_{k=0}^{2^n-1} \frac{\partial^2 f}{\partial t \partial x} \Delta t_k \Delta X_{t_k}}_{\to 0} + \underbrace{\frac{1}{2} \sum_{k=0}^{2^n-1} \frac{\partial^2 f}{\partial t^2} (\Delta t_k)^2}_{\to 0} + \underbrace{\sum_{k=0}^{2^n-1} R_k}_{\to 0}. \end{aligned}$$

For $f(t,x) = \frac{1}{2}x^2$ with $X_t = W_t$ and $a_t = 0, b_t = 1$. Applying Itô formula:

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + a_t\frac{\partial f}{\partial x}dt + b_t\frac{\partial f}{\partial x}dW + \frac{1}{2}b_t^2\frac{\partial^2 f}{\partial x^2}dt \\ &= \frac{\partial f}{\partial t}dt + (1)\frac{\partial f}{\partial x}dW + \frac{1}{2}(1)^2\frac{\partial^2 f}{\partial x^2}dt \\ &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dW + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt \\ &= \frac{\partial f}{\partial x}dW + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt \end{aligned}$$

Hence $\frac{1}{2}dW_t^2 = W_t dW_t + \frac{1}{2}dt$ and $\frac{1}{2}\int dW_s^2 = \int W_s dW_s + \frac{1}{2}\int dt$ Thus,

$$\boxed{I_t = \int_0^t W_s dW_s = \frac{1}{2}(W_t^2 - t)}$$

We got the same result as by using the variational calculus.

Let a (drift) and b (diffusion) are two measurable vector functions defined as follows: $a : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $b : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$. If the $d-$dim SDE with a F_{t_0}-measurable initial condition $X_{t_0} = \Upsilon$ (RV), independent from $\sigma(W)$, satisfies:

1. (Lipschitz condition) For $t \in [t_0, T]$, $x, y \in \mathbb{R}^d$ and a constant $K > 0$:

$$\|a(t, x) - a(t, y)\| \leqslant K\|x - y\| \quad \text{and} \quad \|b(t, x) - b(t, y)\| \leqslant K\|x - y\|$$

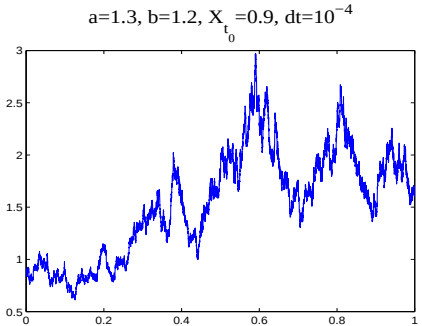
2. (Linear Growth) For $t \in [t_0, T]$, $x \in \mathbb{R}^d$ and a constant $D > 0$:

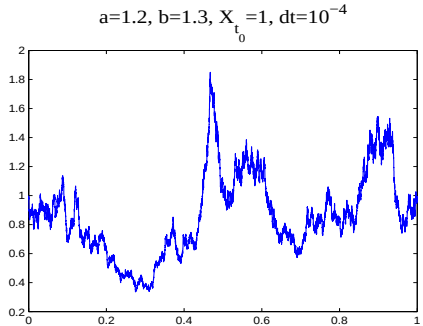
$$\|a(t, x)\|^2 + \|b(t, x)\|^2 \leqslant D(1 + \|x\|^2).$$

then the SDE has a **pathwise unique solution** X_t on $[t0, T]$ with $E\left(\sup_{t_0\leqslant s\leqslant T} |X_s|^2\right) < \infty$.

Consider the linear SDE: $X_t = aX_t ds + bX_t dW_t$ with exact solution

$$X_t = X_{t_0} \exp\left((a - \frac{1}{2}b^2)t + bW_t\right)$$





Itô – Integral

$$\frac{1}{2}((\Delta W_{t_i})^2 - \Delta t_i) = \int_{t_i}^{t_{i+1}} W_s dW_s$$

Gaussian (White Noise)

$$\frac{d}{dt}W_t = \xi_t$$

$$\xi_t \sim \frac{1}{\sqrt{\Delta t}}\mathcal{N}(0,1)$$

W_t

BM increment

$$\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}$$

$$\Delta W_{t_i} \sim \mathcal{N}(0, \sqrt{\Delta t_i})$$

Gaussian (Random Variable)

$$\int_{t_i}^{t_{i+1}} W_s ds = Z_{\Delta_{t_i}}$$

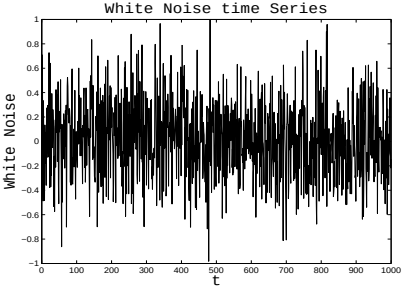
$$Z_t \sim \sqrt{\frac{t^3}{3}}\mathcal{N}(0,1)$$

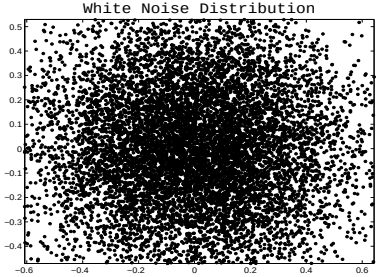
The white noise as a stationary process has the following properties:

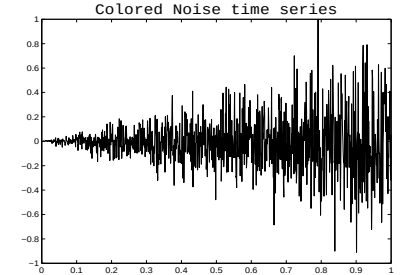
$$E(\xi_{\Delta t}) = 0; \quad Var(\xi_{\Delta t}) = \frac{1}{\Delta t}; \; Cov\Big(\xi_{\Delta t}(t), \xi_{\Delta s}(s)\Big) = 0; \quad \text{if } t \neq s$$

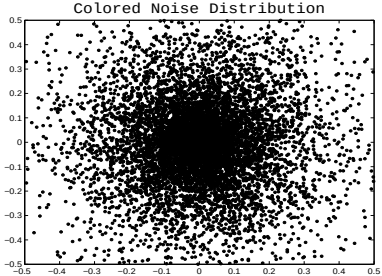
This noise is called white whenever one talks about uncorrelated (or independent) noise at each pixel. White noise is the noise signal whose power spectrum is flat (the Fourier transform of its covariance). Otherwise the noise is called Colored Noise.

Time series of White and colored Noises









Consider the RODE

$$\boxed{\frac{dx}{dt} = a(x) + b(x)\zeta_t^{\Delta}} \tag{14}$$

- The noise in physical systems usual has a wide band spectrum, i.e. a Δ-correlated stationary Gaussian process ζ_t^{Δ}, white noise limit as the time step $\Delta \to 0$.
- The RODE (14) leads to the so called Stratonovich[1] SDE

$$\boxed{dX_t = a(X_t) + b(X_t) \circ dW_t} \tag{15}$$

- The Stratonovich SDE with a Wiener process can be rewritten as stochastic differential system of equations

EXAMPLE

$$\frac{dx}{dt} = -x + \sin(W_t) \Leftrightarrow d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} -X_t + \sin(Y_t) \\ 0 \end{pmatrix} dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dW_t$$

[1] *Wong & Zakai (1965), Godin & Molchanov (2007)*

Part II

Numerical Schemes for SDEs

- *Itô Taylor-Expansion*
- *Itô Taylor Family Schemes (Strong and Weak convergence)*
- *Analysis of Itô Taylor Schemes*

Differential equations with noise Langevin used 'noisy' differential equations to model the particle dynamics of Brownian motion in the early 1900s:

$$\boxed{\frac{dx}{dt} = a(t,x) + b(t,x)\eta_t, \qquad \text{noise} : \eta_t} \tag{16}$$

Gaussian white noise: many mathematical problems finally resolved by Itô in the 1940s with the introduction of stochastic calculus
$\Rightarrow$ stochastic differential equations (SDEs)

$$\boxed{\frac{dx}{dt} = a(t,x) + b(t,x)dW_t, \qquad \text{Wiener process} : W_t} \tag{17}$$

more regular noise $\Rightarrow$ random ordinary differential equations (RODEs)

For $Y_t = f(t, X_t)$ consider the one dimensional SDE

$$dY_t = a(t, X_t)dt + b(t, X_t)dW_t$$

The one dimensional Itô Formula is given by

$$\boxed{dY_t = L^0 f(t, X_t)dt + L^1 f(t, X_t)dW_t,} \tag{18}$$

Where the differential operators L^0 and L^1 are given as follows:

$$L^0 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \underbrace{\frac{1}{2}b^2\frac{\partial^2}{\partial x^2}}_{\text{due to } E(\Delta W_t)=\Delta t} \qquad \text{and} \qquad L^1 = b\frac{\partial}{\partial x}.$$

Given the Itô Formula for $Y_t = f(t, X_t)$:

$$\begin{aligned} dY_t &= \left\{ \frac{\partial f}{\partial t} + \sum_{k=1}^{d} a_t^k \frac{\partial f}{\partial x^k} + \frac{1}{2} \sum_{k,i=1}^{d} \sum_{j=1}^{m} b^{i,j} b^{k,j} \frac{\partial^2 f}{\partial x^i \partial x^k} \right\} dt \\ &+ \sum_{j=1}^{m} \sum_{i=1}^{d} b^{i,j} \frac{\partial f}{\partial x^i} dW_t^j \\ &= L^0 f + \sum_{j=1}^{m} L^j f \, dW_t^j, \end{aligned}$$

where for $i = 1, \ldots, d$ and $j = 1, \ldots, m$ the Itô coefficient functions are:

$$\begin{aligned} L^0 &= \frac{\partial}{\partial t} + \sum_{k=1}^{d} a_t^k \frac{\partial}{\partial x^k} + \frac{1}{2} \sum_{k,i=1}^{d} \sum_{j=1}^{m} b^{i,j} b^{k,j} \frac{\partial^2}{\partial x^i \partial x^k}, \\ L^j &= \sum_{i=1}^{d} b^{i,j} \frac{\partial}{\partial x^i}. \end{aligned}$$

Consider the one dimensional Itô-Process X_t, which satisfy the Itô-Integral equation:

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)ds + \int_{t_0}^{t} b(X_s)dW_s \tag{19}$$

where a and b reel valued smooth functions, satisfying time-growth condition and $f \in C^2(\mathbb{R}, \mathbb{R})$. By applying the Itô Formula for $f(X_s)$, it follows:

$$\begin{aligned} f(X_s) &= f(X_{t_0}) + \int_{t_0}^{t} \left(a(X_s)\frac{\partial}{\partial x} f(X_s) + \frac{1}{2} b^2(X_s)\frac{\partial^2}{\partial x^2} f(X_s) \right) ds \\ &\quad + \int_{t_0}^{t} b(X_s)\frac{\partial}{\partial x} f(X_s) dW_s \\ &= f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s) ds + \int_{t_0}^{t} L^1 f(X_s) dW_s. \end{aligned}$$

If $f(x) = x \in \mathbb{R}$ then $L^0 f = a$ and $L^1 f = b$ the Itô-Taylor Expansion is given in integral form as

$$\begin{aligned} X_t &= X_{t_0} + \int_{t_0}^{t} \left(a(X_{t_0}) + \int_{t_0}^{s} L^0 a(X_z) dz + \int_{t_0}^{s} L^1 a(X_z) dW_z \right) ds \\ &\quad + \int_{t_0}^{t} \left(b(X_{t_0}) + \int_{t_0}^{s} L^0 b(X_z) dz + \int_{t_0}^{s} L^1 b(X_z) dW_z \right) dW_s \\ &= X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s + \underbrace{R_1}_{Remainder} \end{aligned}$$

where the remainder term is given by:

$$\begin{aligned} R_1 &= \int_{t_0}^{t} \int_{t_0}^{s} L^0 a(X_z) dz ds + \int_{t_0}^{t} \int_{t_0}^{s} L^1 a(X_z) dW_z ds \\ &\quad + \int_{t_0}^{t} \int_{t_0}^{s} L^0 b(X_z) dz dW_s + \int_{t_0}^{t} \int_{t_0}^{s} \underbrace{L^1 b(X_z)}_{Expand\ next} \, dW_z dW_s. \end{aligned}$$

By applying the Itô-Formula to the function $L^1 b$, we get(Milstein schemes)

$$\begin{aligned} X_t &= X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s \\ &\quad + L^1 b(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dW_z dW_s + \underbrace{R_2}_{Remainder} \end{aligned}$$

where the remainder term is given by:

$$\begin{aligned} R_2 &= \int_{t_0}^{t} \int_{t_0}^{s} L^0 a(X_z) dz ds + \int_{t_0}^{t} \int_{t_0}^{s} L^1 a(X_z) dW_z ds \\ &\quad + \int_{t_0}^{t} \int_{t_0}^{s} L^0 b(X_z) dz dW_s + \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^0 L^1 b(X_u) du dW_z dW_s \\ &\quad + \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^1 \underbrace{L^1 b(X_u)}_{Expand\ next} dW_u dW_z dW_s. \end{aligned}$$

Don't forget $dtdt = dtdW_t = dW_t dt = 0$ and $dW_t dW_t = dt$.

A set $\mathcal{A}$ of Multi-Index is called **Hierarchical Index-Set**, if the following condition are satisfied

- $\mathcal{A} \neq \emptyset$,
- $\ell_m = \sup_{\alpha \in \mathcal{A}} \quad l(\alpha) < \infty$,
- $-\alpha \in \mathcal{A}; \quad \forall\, \alpha \in \mathcal{A} \backslash \{v\}$.

For any hierarchical set $\mathcal{A}$, we define the corresponding **Remainder Index-Set** as:

$$\mathcal{B}(\mathcal{A}) = \{\alpha \in \mathcal{M} \backslash \{\mathcal{A}\} : -\alpha \in \mathcal{A}\}.$$

Theorem

Let ρ and τ are two Stop-times such that $t_0 \leqslant \rho(w) \leqslant \tau(w) \leqslant T$. If the function $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}$ and all its differential are smooth and if the partial differential of a and b exist, then for any set $\mathcal{A} \in \mathcal{M}$ follow the stochastic Taylor-Expansion

$$f(\tau, X_\tau) = \sum_{\alpha \in \mathcal{A}_\gamma} I_\alpha[f_\alpha(\rho, X_\rho)] + \sum_{\alpha \in \mathcal{B}(\mathcal{A}_\gamma)} I_\alpha[f_\alpha(., X.)]_{\rho,\tau}, \tag{20}$$

Where the **Itô coefficient functions** are defined by

$$f_\alpha = \begin{cases} f & \text{if} \quad \ell(\alpha) = 0; \\ L^{j_1} f_{-\alpha} & \text{if} \quad \ell(\alpha) > 0. \end{cases}$$

for $\alpha = (j_1, \ldots, j_l) \in M$ and $f \in C^{n(\alpha), \ell(\alpha)+n(\alpha)}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R})$.

In the EM scheme, the numerical solution to the SDE, Y_t converges to the exact solution X_t. We have to distinct between the following types of convergence :

- (**Path Approximation**). We said that the c**onvergence is strong** of order γ if
$$\lim_{N_T \to \infty} E\, s(\gamma) := \lim_{N_T \to \infty} E\left(|Y^{\gamma}_{N_T} - X_t|\right) = 0.$$
- (**Distributional Approximation**). We said that the **convergence is weak** of order γ if
$$\lim_{N_T \to \infty} E\, w(\gamma) := \lim_{N_T \to \infty} \left| E\, g\left(Y^{\gamma}_{N_T}\right) - E\, g(X_t) \right| = 0.$$

Where N_T refers the number of the time steps used for the approximation, g any polynomial function.

For $l \in \mathbb{N}$ and $\alpha = (j_1, \ldots, j_l)$ it exists a positive constant K independent from Δ such that

$$\boxed{\left(\boldsymbol{E}(I^2_{\alpha,\Delta})\right)^{\frac{1}{2}} \leqslant K\Delta^{\frac{\ell(\alpha)+n(\alpha)}{2}}} \tag{21}$$

Moreover, for X a d-dim stoch. process sol. of SDE, a multi-Index $\alpha \in \mathcal{M}$ and $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}$ satisfying If $|f_\alpha(t,x)|^2 \leqslant K\left(1 + |x|^2\right)$, the we have

$$\boxed{\boldsymbol{E}\left|I_\alpha\left[f_\alpha(t, X_t)\right]_{0,\Delta}\right|^2 \leqslant K\left(1 + \boldsymbol{E}|X_t|^2\right)\Delta^{l(\alpha)+n(\alpha)}} \tag{22}$$

Remark that the real number $\frac{\ell(\alpha)+n(\alpha)}{2} = \gamma$ or $\ell(\alpha) + n(\alpha) = \gamma$ could be used as a convergence order of the Itô-Taylor approximation of SDE solutions.

- **So, we have to characterize the sets $\mathcal{A}_\gamma$ and $\mathcal{B}_\gamma(\mathcal{A}_\gamma)$.**
- **Two types of convergence orders !!!**

THEOREM

Consider the strong Itô-Taylor approximation $Y^{\Delta} = \left\{Y_t^{\Delta},\ t \in]t_0, T[\right\}$ *with convergence order* $\gamma = 0.5, 1, 1.5, \ldots$. *If*

- $\boldsymbol{E}(|X_{t_0}|^2) < \infty \quad$ *and* $\quad \left(\boldsymbol{E}(|X_{t_0} - Y_{t_0}^{\Delta}|^2)\right)^{\frac{1}{2}} \leqslant K_1 \Delta^{\gamma}$
- $|f_\alpha(t,x) - f_\alpha(t,y)| \leqslant K_2|x-y| \qquad \forall \alpha \in \mathcal{A}_\gamma,\ t \in [t_0, T]$ *and* $x, y \in \mathbb{R}^d$,
- $f_{-\alpha} \in C^{1,2} \quad$ *and* $\quad f_\alpha \in H_\alpha \qquad \forall \alpha \in \mathcal{A}_\gamma \cup \mathcal{B}_\gamma$,
- $|f_\alpha(t,x)| \leqslant K_3(1+|x|) \qquad \forall \alpha \in \mathcal{A}_\gamma \cup \mathcal{B}_\gamma, t \in [t_0, T]$ *and* $x \in \mathbb{R}^d$.

then it holds

$$\boldsymbol{E}\left(|X_T - Y_T^{\Delta}|\right) \leqslant K_4 \Delta^{\gamma}. \tag{23}$$

where $K_i, i = 1, 2, 3, 4.$ *are real pairwise independent positive constants .*

For the proof we refer to Kloeden-Platten p. 361ff)

On the Convergence Theorem Assumptions

The convergence theorem assumes that

- the coefficient functions f_α in the Taylor scheme are uniformly bounded on $\mathbb{R}^d$: i.e., the partial derivatives of appropriately high order of the SDE coefficient functions $a, b^1, \ldots, b^m$ are uniformly bounded on R^d.
- However, this assumption is **not satisfied in many important applications**:

1. **Cox-Ingersoll-Ross models** in finance with $V_t \in [0, 1]$

$$dV_t = \alpha(\beta - V_t)dt + \mu\sqrt{V_t}dW_t$$

2. **Duffing-van der Pol oscillator** with multiplicative noise:

$$\begin{aligned} dX_t^1 &= X_t^2 dt \\ dX_t^2 &= \Big(- X_t^1 + \beta X_t^2 - (X_t^1)^3 - (X^1 t)^2 X^2 t \Big) dt + \sigma X_t^2 dW_t \end{aligned}$$

Proofs in the literature (e.g. in the monographs of G. Milstein)

Construction of Multi-Index Sets in the Itô-Taylor-Expansion

For $\gamma = 0.5, 1, 1.5, 2, \ldots$ the set of Multi-indices $\mathcal{A}_\gamma$ is called Hierarchical Index-Set if

$$\mathcal{A}_\gamma = \{\alpha \in \mathcal{M} : \ \ell(\alpha) + n(\alpha) \leqslant 2\gamma \text{ or } \ell(\alpha) = n(\alpha) = \gamma + \frac{1}{2}\}.$$

- Remark that the three assumption of a Hierarchical set are satisfied.
- The construction of the corresponding remainder set is given by

$$\mathcal{B}(\mathcal{A}_\gamma) = \{\alpha \in \mathcal{M} \backslash \mathcal{A}_\gamma : -\alpha \in \mathcal{A}_\gamma\}.$$

- (Later) γ will refer the convergence order the Itô-Taylor family schemes.

For $F = \{0, 1\}$, we have

- If $\ell_m = 0$ (i.e $\gamma = 0$) then $\mathcal{A}_0 = \{v\}$ and $\mathcal{B}(A_0) = \{(0), (1)\}$.
- If $\ell_m = 1$ (i.e $\gamma = 0.5$) then $\mathcal{A}_{0.5} = \{v, (0), (1)\}$ and

$$\mathcal{B}(\mathcal{A}_{0.5}) = \{(0, 0), (1, 0), (0, 1), (1, 1)\}.$$

- If $\ell_m = 2$ (i.e $\gamma = 1$) then $\mathcal{A}_1 = \{v, (0), (1), (1, 1)\}$ and

$$\mathcal{B}(\mathcal{A}_1) = \{(0, 0), (1, 0), (0, 1), (0, 1, 1), (1, 1, 1)\}.$$

- If $\ell_m = 3$ (i.e $\gamma = 1.5$) then $\mathcal{A}_{1.5} = \{v, (0), (1), (1, 1), (0, 0), (1, 0), (0, 1), (1, 1, 1)\}$ and $\mathcal{B}(\mathcal{A}_{1.5}) =$
$\{(0, 0, 0), (1, 0, 0), (1, 1, 0), (0, 1, 0), (1, 0, 1), (0, 0, 1), (0, 1, 1), (0, 1, 1, 1), (1, 1, 1, 1)\}$.

$$\mathcal{A}_{0.0} = \{v\}$$

$$\gamma = 0.5 \quad \text{Euler-Maruyama}$$
$$\mathcal{A}_{0.5} = \{v, (0), (1)\}$$

$$\gamma = 1. \quad \text{Milstein/ Taylor}$$
$$\mathcal{A}_{1.0} = \{v, (0), (1), (1,1)\}$$

$$\gamma = 1.5 \quad \text{Taylor}$$
$$\mathcal{A}_{1.5} = \{v, (0), (1), (1,1), (0,0), (1,0), (0,1), (1,1,1)\}$$

$$\gamma = 2 \quad \text{Taylor}$$
$$\mathcal{A}_{2.0} = \Big\{v, (0), (1), (1,1), (0,0), (1,0), (0,1), (1,1,1),$$
$$(1,1,0), (0,1,1), (1,0,1), (0,1,1), (1,1,1,1)\Big\}$$

The strong Itô-Taylor scheme of order $\gamma = 0.5,\ 1,\ 1.5,\ 2, \ldots$ is given by

$$\boxed{Y_{n+1} = Y_n + \sum_{\alpha \in \mathcal{A}_\gamma \setminus \{v\}} I_\alpha[\, f_\alpha(t_n, Y_n)]_{t_n, t_{n+1}}} \tag{24}$$

for $n = 0, 1, 2, \ldots$, and the initial value $Y_0 = \xi_0$.
$\mathcal{M} = \{v\} \bigcup \{\alpha = (j_1, j_2, \ldots, j_l);\ j_i \in F;\ l \in \{1, 2, \ldots\}\}$.

Consider the autonomous case $dX_t = a(X_t)dt + b(X_t)dW_t$.
An example of Itô-Taylor scheme for a d dimensional Itô-Process X_t and an index set $\mathcal{M} = \{1, \ldots, m\}$ (i.e m dimensional BM) is given by:

$$Y_{n+1} = Y_n + I_v[a(Y_n)]_{t_n,t_{n+1}} + \sum_{j=1}^{m} I_{(j)}[b^j(Y_n)]_{t_n,t_{n+1}} \tag{25}$$

$$+ \sum_{j_1,j_2=1}^{m} I_{(j_1,j_2)}[L^{j_1}b^{j_2}(Y_n)]_{t_n,t_{n+1}} \tag{26}$$

$$+ \sum_{j=1}^{m} I_{(j,0)}[L^j a(Y_n)]_{t_n,t_{n+1}} + \sum_{j=1}^{m} I_{(0,j)}[L^0 b^j(Y_n)]_{t_n,t_{n+1}} \tag{27}$$

$$+ \sum_{j=1}^{m} I_{(0,0)}[a(Y_n)]_{t_n,t_{n+1}} + \sum_{j,j_1,j_2=1}^{m} I_{(j,j_1,j_2)}[L^j L^{j_1} b^{j_2}(Y_n)]_{t_n,t_{n+1}}. \tag{28}$$

Euler Maruyama (25), Milstein (25)-(57) and Taylor of order 1.5 is represented by (25)-(28).

$$
\begin{aligned}
I_v[a(X_n)]_{t_n,t_{n+1}} &= a(X_n)\Delta t_n \\
I_{(1)}[b^1(X_n)]_{t_n,t_{n+1}} &= b^1(X_n)\int_{t_n}^{t_{n+1}} dW_t^1 \\
&= b(X_n)\Delta W_{t_n} \\
I_{(1,1)}[L^1 b^1(t_n,X_n)]_{t_n,t_{n+1}} &= b^1\frac{\partial b^1}{\partial x}(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW_s^1 dW_t^1 \\
&= b(X_n)\frac{\partial b}{\partial x}(X_n)\frac{1}{2}\left((\Delta W_{t_n})^2 - \Delta t_n\right) \\
I_{(1,0)}[L^1 a(X_n)]_{t_n,t_{n+1}} &= b^1(X_n)\frac{\partial a}{\partial x}(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW_s^1 dt \\
&= b(X_n)\frac{\partial a}{\partial x}(X_n)\Delta Z_{t_n} \qquad \Delta Z_{t_n} \sim N\left(0, \frac{1}{3}(\Delta t_n)^3\right)
\end{aligned}
$$

Recall $\quad L^0 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \frac{1}{2}b^2\frac{\partial}{\partial x^2}, \qquad L^1 = b\frac{\partial}{\partial x}.$

$$
\begin{aligned}
I_{(0,1)}[L^0 b^1(X_n)]_{t_n,t_{n+1}} &= \left(a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\right)(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdW_t \\
&= \left(a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\right)(X_n)\left(\Delta t_n \Delta W_{t_n} - \Delta Z_{t_n}\right) \\
I_{(0,0)}[L^0 a(X_n)]_{t_n,t_{n+1}} &= L^0 a(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdt \\
&= \left(a\frac{\partial a}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 a}{\partial x^2}\right)(X_n)\frac{1}{2}(\Delta t_n)^2 \\
I_{(1,1,1)}[L^1 L^1 b^1(X_n)]_{t_n,t_{n+1}} &= L^1 L^1 b\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\int_{t_n}^{s} dW_r dW_s dW_t \\
&= \left(b\left(\frac{\partial b}{\partial x}\right)^2 + b^2\frac{\partial^2 b}{\partial x^2}\right)(X_n)\left(\frac{(\Delta W_{t_n})^2}{6} - \frac{\Delta t_n}{2}\right)\Delta W_{t_n}
\end{aligned}
$$

Strong Euler-Maruyama Scheme (1955), $\gamma = 0.5$

Consider the **Geometric Brownian Motion**, which satisfies the SDE

$$\boxed{\begin{aligned} X_0 &= \xi \\ dX_t &= \lambda X_t dt + \mu X_t dW_t \end{aligned}}$$

where λ is called the percentage drift and μ is called the percentage volatility. The exact solution is

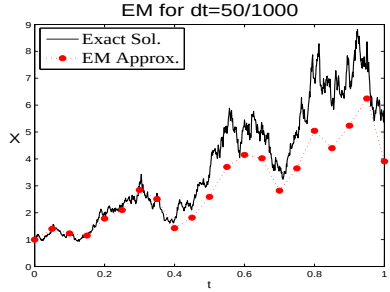
$$X_t = X_0 \exp\left((\lambda - \frac{\mu^2}{2})t - \mu W_t\right)$$

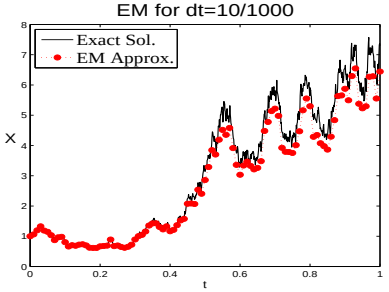
Numerical solution: for $\mathcal{A}_{0.5} = \{v, (0), (1)\}$ (i.e $\ell_1 = 1$), we define the **Euler-Maruyama** scheme

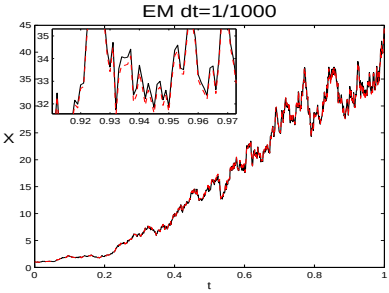
$$\boxed{\begin{aligned} X_0 &= Y_0 = \xi \\ Y_{n+1} &= Y_n + \lambda Y_n \Delta t_n + \mu Y_n \Delta W_{t_n} \end{aligned}}$$

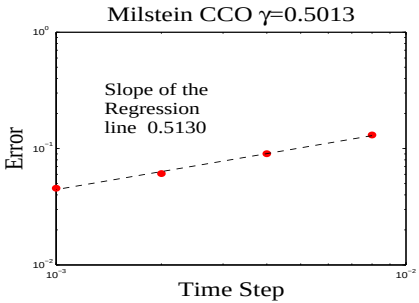
Where Y_n in the numerical approximation of the exact solution X_{t_n}.

Numerical Simulation of GBM by using EM.









Where $dt = 10^{-3}$ is used to simulate exact solution.

Consider the SDE

$$\boxed{\begin{aligned} X_0 &= \xi \\ dX_t &= a(t, X_t)dt + b(t, X_t)dW_t \end{aligned}}$$

Numerical solution: for $\mathcal{A}_{0.5} = \{v, (0), (1)\}$ (i.e $\ell_1 = 1$), we define the **Euler-Maruyama** scheme

$$\boxed{\begin{aligned} X_0 = Y_0 &= \xi \\ Y_{n+1} &= Y_n + a(t_n, Y_n)\Delta t_n + b(t_n, Y_n)\Delta W_{t_n} \end{aligned}}$$

Where Y_n in the numerical approximation of the exact solution X_{t_n} and

$$\begin{aligned} \Delta t_n &= \int_{t_n}^{t_{n+1}} dt \\ \Delta W_{t_n} &= \int_{t_n}^{t_{n+1}} dW_s. \end{aligned}$$

For $\mathcal{A}_1 = \{v, (0), (1), (1,1)\}$ (i.e $\ell_1 = 2$), we define the **Milstein** scheme

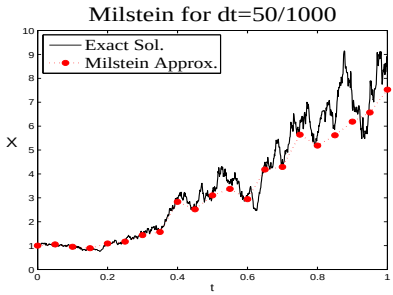
$$\boxed{\begin{aligned} X_0 &= Y_0 = \xi \\ Y_{n+1} &= Y_n + a(t_n, Y_n)\Delta t_n + b(t_n, Y_n)\Delta W_{t_n} \\ &\quad + \frac{1}{2} b(t_n, Y_n)\frac{\partial b}{\partial x}(t_n, Y_n))((\Delta W_{t_n})^2 - \Delta t_n) \end{aligned}} \tag{29}$$

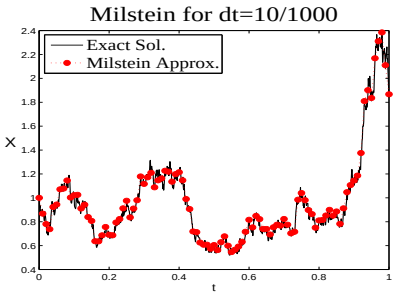
Where Y_n in the numerical approximation of the exact solution X_{t_n} and the additional (to E.M) is

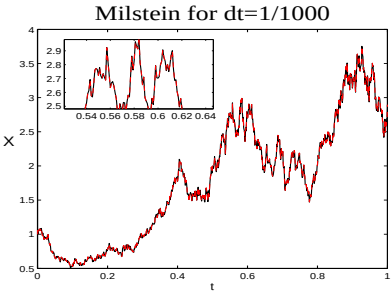
$$f_{(1,1)} I_{(1,1)} \quad = \quad L^1 b(t_n, Y_n) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s^1 dW_t^1 .$$

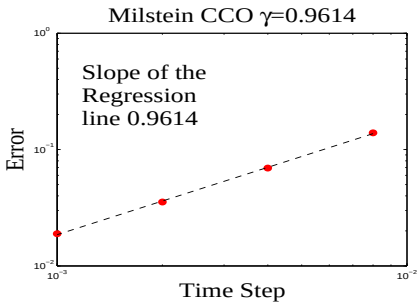
The Milstein method is a first order Itô-Taylor method. It is derived from a truncation of the stochastic Itô-Taylor expansion of the solution.

Numerical Simulation of GBM by using Milstein.









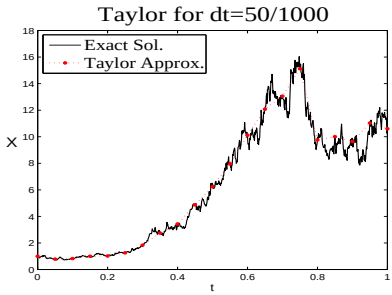
Where $dt = 10^{-3}$ is used to simulate exact solution.

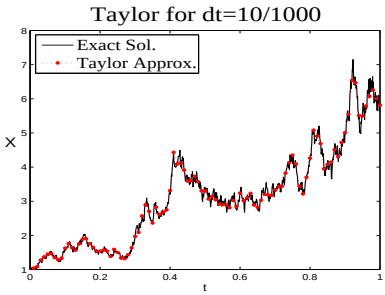
For $\mathcal{A}_{1.5} = \{v, (0), (1), (1,1), (0,0), (1,0), (0,1), (1,1,1)\}$ (i.e $\ell_1 = 3$), we define the **Itô-Taylor** scheme with order $\gamma = 1.5$:

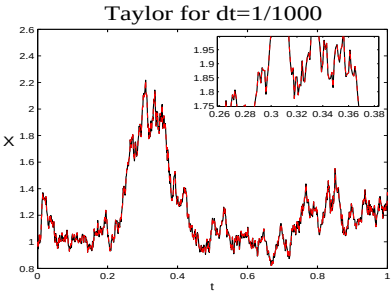
$$
\begin{aligned}
X_0 &= Y_0 = \xi \\
Y_{n+1} &= Y_n + a\Delta t_n + b\Delta W_{t_n} + \frac{1}{2} b \frac{\partial b}{\partial x}((\Delta W_{t_n})^2 - \Delta t_n) \\
&\quad + \frac{1}{2}\left(a\frac{\partial a}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 a}{\partial x^2}\right)(\Delta t)^2 + b\frac{\partial a}{\partial x}\Delta Z_{t_n} \\
&\quad + \left(a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\right)\left(\Delta t_n \Delta W_{t_n} - \Delta Z_{t_n}\right) \\
&\quad + \frac{1}{2}\left(b\left(\frac{\partial b}{\partial x}\right)^2 + b^2\frac{\partial^2 b}{\partial x^2}\right)\left(\frac{1}{3}(\Delta W_{t_n})^2 - \Delta t_n\right)\Delta W_{t_n}
\end{aligned}
\tag{30}
$$

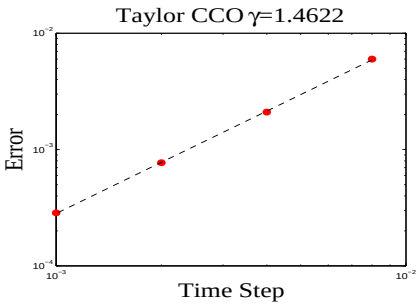
Where Y_n in the numerical approximation of the exact solution X_{t_n} and ΔZ_{t_n} is a normal distributed random variable $\Delta Z_{t_n} \sim N(0, \frac{1}{3}(\Delta t_n)^3)$

Numerical Simulation of GBM by using Taylor.

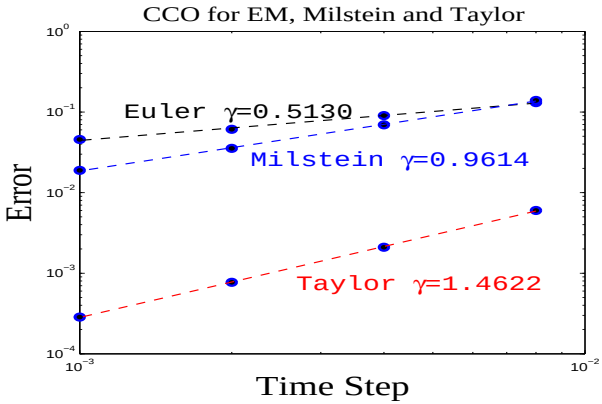








Where $dt = 10^{-3}$ is used to simulate exact solution.



Increasing the number of discretization time-points and the number of simulations leads to the best estimation of computational convergence orders. In our cases, we have used a time step $dt = 10^{-3}$ and $M = 1000$ simulation for each numerical simulation.

In the Milstein scheme, the replacement of the first derivative of the diffusion b_x by

$$\frac{\partial b}{\partial x}(X_{t_n}) \approx \frac{b\big(X_{t_n} + b(X_{t_n})\sqrt{\Delta t_n}\big) - b\big(X_{t_n}\big)}{b\big(X_{t_n}\big)\sqrt{\Delta t_n}}$$

leads to Strong Order 1.0 **Runge-Kutta** Method

$$\begin{aligned} X_0 &= Y_0 = \xi \\ Y_{n+1} &= Y_n + a(Y_n)\Delta t_n + b(Y_n)\Delta W_{t_n} \\ &\quad + \frac{1}{2}\left(b\big(Y_n + b(Y_n)\sqrt{\Delta t_n}\big) - b\big(Y_n\big)\right)\frac{(\Delta W_{t_n})^2 - \Delta t_n)}{\sqrt{\Delta t_n}} \end{aligned} \tag{31}$$

For $\beta = 1, 2, \ldots$ the set of Multi-indices $\mathcal{A}_\beta^w$ is called Hierarchical Index-Set for the **weak approximation** if

$$\mathcal{A}_\beta^w = \{\alpha \in \mathcal{M} : \ \ell(\alpha) \leqslant \beta\}.$$

The weak approximation of the solution of a SDE is given by

$$\boxed{\begin{aligned} X_0 &= Y_0 = \xi \\ Y_n &= \sum_{\alpha \in \mathcal{A}_\beta^w} I_\alpha[\, f_\alpha(t_n, Y_{t_n})] \end{aligned}} \tag{32}$$

Convergence with weak order β requires the regularity of the drift vector-function a and the regularity of the entries of the diffusion matrix b. This is necessary to ensure that f_α are Lipschitz and have bounded linear growth.

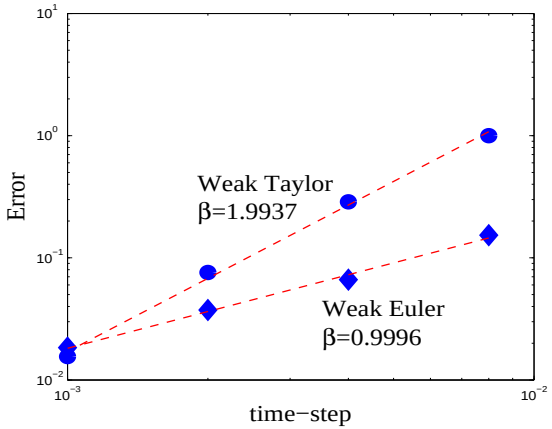
For $\mathcal{A}_1 = \{v, (0), (1)\}$ (i.e $\ell_1 = 1$) (the same as by the strong scheme), we define the **Weak Euler-Maruyama** scheme with order $\beta = 1$

$$\boxed{\begin{aligned} X_0 &= Y_0 = \xi \\ Y_{n+1} &= Y_n + a(Y_n)\Delta t_n + b(Y_n)\Delta W_{t_n} \end{aligned}} \tag{33}$$

Where Y_n in the numerical approximation of the exact solution X_{t_n}.
For $\mathcal{A}_2 = \{v, (0), (1), (0,0), (0,1), (1,0), (1,1)\}$ (i.e $\ell_1 = 2$), we define the **Weak Taylor** with order $\beta = 2$

$$\boxed{\begin{aligned} X_0 &= Y_0 = \xi \\ Y_{n+1} &= Y_n + a\Delta t_n + b\Delta W_{t_n} + \frac{1}{2} b \frac{\partial b}{\partial x}((\Delta W_{t_n})^2 - \Delta t_n) \\ &\quad + \frac{1}{2}\Big(a\frac{\partial a}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 a}{\partial x^2}\Big)(\Delta t)^2 + b\frac{\partial a}{\partial x}\Delta Z_{t_n} \\ &\quad + \Big(a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\Big)\Big(\Delta t_n \Delta W_{t_n} - \Delta Z_{t_n}\Big) \end{aligned}} \tag{34}$$

CCO FOR WEM AND WT



Given the two dimensional system of SDE

$$dX_t^k = a^k(t, X_t)dt + \sum_{j=1}^{2} b^{k,j}(t, X_t)dW_t^j, \qquad X_0^k = \xi^k \tag{35}$$

For $F = \{0, 1, 2\}$ and $\mathcal{A}_1 = \{v, (0), (1), (2), (1, 1), (1, 2), (2, 2), (2, 1)\}$, we define the **two dimensional Milstein** scheme as

$$\begin{aligned} Y_{n+1}^1 &= Y_n^1 + a^1\Delta t_n + b^{1,1}\Delta W_{t_n}^1 + b^{1,2}\Delta W_{t_n}^2 \\ &+ \left(b^{1,1}\frac{\partial b^{1,1}}{\partial x^1} + b^{2,1}\frac{\partial b^{1,1}}{\partial x^2}\right) I_{(1,1)} + \left(b^{1,1}\frac{\partial b^{1,2}}{\partial x^1} + b^{2,1}\frac{\partial b^{1,2}}{\partial x^2}\right) I_{(1,2)} \\ &+ \left(b^{1,2}\frac{\partial b^{1,1}}{\partial x^1} + b^{2,2}\frac{\partial b^{1,1}}{\partial x^2}\right) I_{(2,1)} + \left(b^{1,2}\frac{\partial b^{1,2}}{\partial x^1} + b^{2,2}\frac{\partial b^{1,2}}{\partial x^2}\right) I_{(2,2)}, \end{aligned}$$

$$\begin{aligned} Y_{n+1}^2 &= Y_n^2 + a^2\Delta t_n + b^{2,1}\Delta W_{t_n}^1 + b^{2,2}\Delta W_{t_n}^2 \\ &+ \left(b^{1,1}\frac{\partial b^{2,1}}{\partial x^1} + b^{2,1}\frac{\partial b^{2,1}}{\partial x^2}\right) I_{(1,1)} + \left(b^{1,1}\frac{\partial b^{2,2}}{\partial x^1} + b^{2,1}\frac{\partial b^{2,2}}{\partial x^2}\right) I_{(1,2)} \\ &+ \left(b^{1,2}\frac{\partial b^{2,1}}{\partial x^1} + b^{2,2}\frac{\partial b^{2,1}}{\partial x^2}\right) I_{(2,1)} + \left(b^{1,2}\frac{\partial b^{2,2}}{\partial x^1} + b^{2,2}\frac{\partial b^{2,2}}{\partial x^2}\right) I_{(2,2)} \end{aligned}$$

For the numerical computation of $I_{(j_1,j_2),\Delta t_n}$, we have to distinct:

Case 1. $(j_1 = j_2 = j)$

$$I_{(j,j),\Delta t_n} = \frac{1}{2}(I^2_{(j),\Delta t} - \Delta t_n) = \frac{1}{2}\Big((\Delta W^j_{t_n})^2 - \Delta t_n\Big). \tag{36}$$

Case 2. $(j_1 \neq j_2)$

$$\begin{aligned} I^p_{(j_1,j_2),\Delta t_n} &= \frac{1}{2}\Delta t_n R_1 R_2 + \Delta t_n \sqrt{\rho_p}(R_3 R_2 - R_4 R_1) \\ &\quad + \frac{\Delta t_n}{2\pi}\sum_{r=1}^{p}\frac{1}{r}(R_5(\sqrt{2}R_2 + R_8) - R_6(\sqrt{2}R_1 + R_7)), \end{aligned} \tag{37}$$

where R_i for $i = 1, \ldots, 8$ (i.i.d) are standard normal distributed random variables and pairwise independent. The coefficient ρ_p could be approximated by

$$\rho_p = \frac{1}{2\pi^2}\sum_{r=p+1}^{\infty}\frac{1}{r^2}.$$

Milstein for Systems of SDE, Simplified Notation

In order to be able to define the **Milstein** for a system of differential equation with n equation, we use the following notation

$$\begin{aligned}
\tilde{L}^1 &= \left(b^{1,1}\frac{\partial}{\partial x^1}, b^{2,1}\frac{\partial}{\partial x^2}\right);\\
\tilde{L}^2 &= \left(b^{1,2}\frac{\partial}{\partial x^1}, b^{2,2}\frac{\partial}{\partial x^2}\right);\\
&\quad \text{for} \quad i,j = 1,2\\
\tilde{b}^{j,\cdot} &= \left(b^{j,1}, b^{j,2}\right);\ \tilde{b}^{\cdot,j} = \left(b^{1,j}, b^{2,j}\right);\ \tilde{b}^{i,j} = \left(b^{i,j}, b^{i,j}\right)\\
\Delta\tilde{W}_{t_n} &= \left(\Delta W^1_{t_n}, \Delta W^2_{t_n}\right)^t.
\end{aligned}$$

Thus the two dimensional Milstein scheme is

$$\begin{aligned}
Y^1_{n+1} &= Y^1_n + a^1\Delta t_n + \tilde{b}^{1,\cdot}\Delta\tilde{W}_{t_n}\\
&\quad + \tilde{L}^1\tilde{b}^{1,1}I_{(1,1)} + \tilde{L}^1\tilde{b}^{1,2}I_{(1,2)}\\
&\quad + \tilde{L}^2\tilde{b}^{1,1}I_{(2,1)} + \tilde{L}^2\tilde{b}^{1,2}I_{(2,2)}.\\
Y^2_{n+1} &= Y^2_n + a^2\Delta t_n + \tilde{b}^{2,\cdot}\Delta\tilde{W}_{t_n}\\
&\quad + \tilde{L}^1\tilde{b}^{2,1}I_{(1,1)} + \tilde{L}^1\tilde{b}^{2,2}I_{(1,2)}\\
&\quad + \tilde{L}^2\tilde{b}^{2,1}I_{(2,1)} + \tilde{L}^2\tilde{b}^{2,2}I_{(2,2)}.
\end{aligned}$$

the **three dimensional Milstein** scheme as

$$
\begin{aligned}
Y^1_{n+1} &= Y^1_n + a^1\Delta t_n + \tilde{b}^{1,\cdot}\Delta\tilde{W}_{t_n} \\
&\quad + \tilde{L}^1\tilde{b}^{1,1}I_{(1,1)} + \tilde{L}^1\tilde{b}^{1,2}I_{(1,2)} + \tilde{L}^1\tilde{b}^{1,3}I_{(1,3)} \\
&\quad + \tilde{L}^2\tilde{b}^{1,1}I_{(2,1)} + \tilde{L}^2\tilde{b}^{1,2}I_{(2,2)} + \tilde{L}^2\tilde{b}^{1,3}I_{(2,3)} \\
&\quad + \tilde{L}^3\tilde{b}^{1,1}I_{(3,1)} + \tilde{L}^3\tilde{b}^{1,2}I_{(3,2)} + \tilde{L}^3\tilde{b}^{1,3}I_{(3,3)} \\
Y^2_{n+1} &= Y^2_n + a^2\Delta t_n + \tilde{b}^{2,\cdot}\Delta\tilde{W}_{t_n} \\
&\quad + \tilde{L}^1\tilde{b}^{2,1}I_{(1,1)} + \tilde{L}^1\tilde{b}^{2,2}I_{(1,2)} + \tilde{L}^1\tilde{b}^{2,3}I_{(1,3)} \\
&\quad + \tilde{L}^2\tilde{b}^{2,1}I_{(2,1)} + \tilde{L}^2\tilde{b}^{2,2}I_{(2,2)} + \tilde{L}^2\tilde{b}^{2,3}I_{(2,3)} \\
&\quad + \tilde{L}^3\tilde{b}^{2,1}I_{(3,1)} + \tilde{L}^3\tilde{b}^{2,2}I_{(3,2)} + \tilde{L}^3\tilde{b}^{2,3}I_{(3,3)} \\
Y^3_{n+1} &= Y^3_n + a^3\Delta t_n + \tilde{b}^{3,\cdot}\Delta\tilde{W}_{t_n} \\
&\quad + \tilde{L}^1\tilde{b}^{3,1}I_{(1,1)} + \tilde{L}^1\tilde{b}^{3,2}I_{(1,2)} + \tilde{L}^1\tilde{b}^{3,3}I_{(1,3)} \\
&\quad + \tilde{L}^2\tilde{b}^{3,1}I_{(2,1)} + \tilde{L}^2\tilde{b}^{3,2}I_{(2,2)} + \tilde{L}^2\tilde{b}^{3,3}I_{(2,3)} \\
&\quad + \tilde{L}^3\tilde{b}^{3,1}I_{(3,1)} + \tilde{L}^3\tilde{b}^{3,2}I_{(3,2)} + \tilde{L}^3\tilde{b}^{3,3}I_{(3,3)}.
\end{aligned}
$$

Lorenz simplified some of the Navier-Stokes equations in the area of fluid dynamics, and obtained the following non-linear three dimensional ordinary differential system:

$$\begin{aligned} \partial_t x &= p(y(t) - x(t)), \\ \partial_t y &= rx(t) - y(t) - x(t)z(t), \\ \partial_t z &= x(t)y(t) - cz(t), \end{aligned} \tag{38}$$

where

- the parameter p is the Prandtl number in a dimensionless parameter of a convecting system that characterizes the regime of convection
- $r = \frac{Ra}{Ra_c}$ is the quotient of the Rayleigh number and critical Rayleigh number
- b is a geometric factor
- Lorenz used the values $p = 10, r = 28, c = \frac{8}{3}$.

The stochastic differential lorenz system (SDE) given as:

$$d\mathbf{X}_t = a(t, \mathbf{X}_t)dt + b(t, \mathbf{X}_t)d\mathbf{W}_t \qquad \mathbf{X}_0 = \xi, \tag{39}$$

where

- $a : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ represents the drift vector,
- $b : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^{m \times d}$ the diffusion matrix
- $(\mathbf{X}_t)_{t \in [t_0, T]}$ is a $d-$dim reel valued stochastic process solution of SDE (**??**).
- $(\mathbf{W}_t)_{t \in [t_0, T]}$ is a three dimensional Brownian motion.

We numerically examine the performance of the Milstein scheme for solving the stochastic Lorenz system. Where

- The drift is the usual deterministic part of the Lorenz system of the differential system of equations (refsdels).
- The diffusion matrix is given as:

$$b : \mathbb{T} \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}, \tag{40}$$

$$b\Big(t, x(t), y(t), z(t)\Big) = \lambda\Big(x(t) + y(t) + z(t)\Big)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Where $\lambda << 1$ is a stabilization factor.

For constructing the numerical Milstein scheme, let us consider the three dimensional index set $F = 0, 1, 2, 3$, then the corresponding Hierarchical set $\mathcal{A}$ is

$$\mathcal{A} = \Big\{v, (0), (1), (2), (3), (1,1), (1,2), (1,3), (2,2), (2,1), (2,3), (3,3), (3,1), (3,2)\Big\}$$

The Milstein scheme can be derived from the stochastic Taylor expansion:

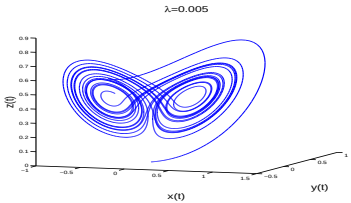
$$Y^1_{n+1} = Y^1_n + a^1(t_n, \mathbf{Y}_n)\Delta + \sum_{j=1}^{3} b^{1,j}(t_n, \mathbf{Y}_n)\Delta W^j_{t_n} + \sum_{j_1,j_2=1}^{3} L^{j_1} b^{1,j_2}(t_n, Y_n) I_{(j_1,j_2)}$$

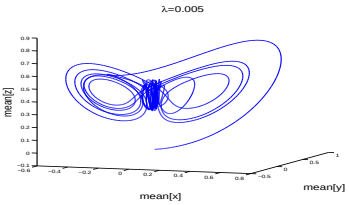
$$Y^2_{n+1} = Y^2_n + a^2(t_n, \mathbf{Y}_n)\Delta + \sum_{j=1}^{3} b^{2,j}(t_n, \mathbf{Y}_n)\Delta W^j_{t_n} + \sum_{j_1,j_2=1}^{3} L^{j_1} b^{2,j_2}(t_n, Y_n) I_{(j_1,j_2)}$$

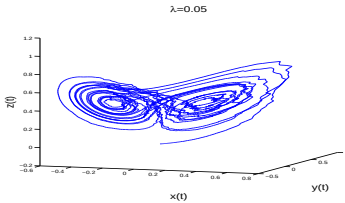
$$Y^3_{n+1} = Y^3_n + a^3(t_n, \mathbf{Y}_n)\Delta + \sum_{j=1}^{3} b^{1,j}(t_n, \mathbf{Y}_n)\Delta W^j_{t_n} + \sum_{j_1,j_2=1}^{3} L^{j_1} b^{3,j_2}(t_n, Y_n) I_{(j_1,j_2)}$$

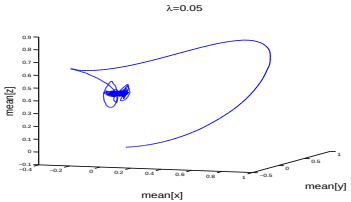
where Δ is a time step of the scheme. We deal here only with a fixed time step. Componentwise, we have

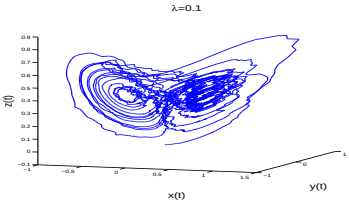
Simulation of the stochastic Lorenz system: The attractor

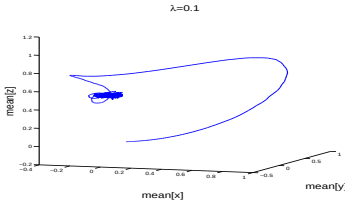






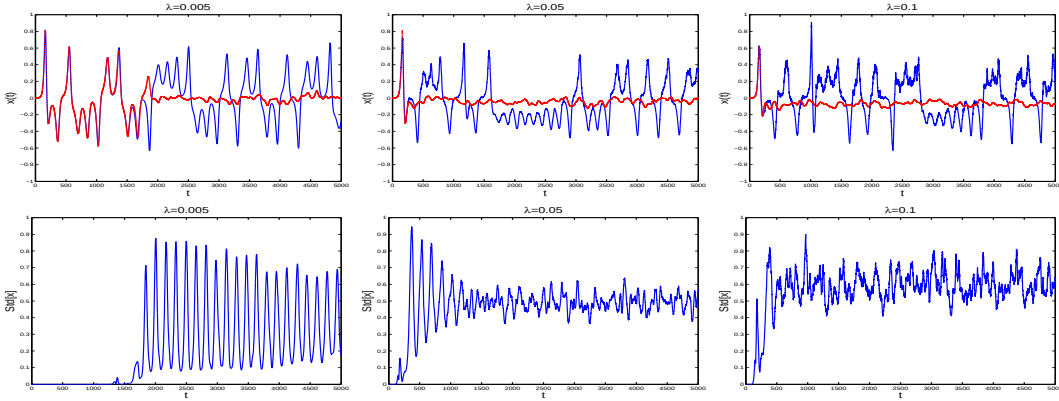






Simulation of the stochastic Lorenz system: Time series $x(t)$

It is important to note that the component of diffusion matrix (40) are correlated. Therefore, the fact that mean behavior of the SLA tends to converge to a stationary state.



(first row) First component ($x(t)$) of SLA (solid line) and the corresponding mean (red line). (second row) the corresponding standard deviation of $x(t)$.

Concluding Remarks

- The Milstein scheme converges with strong order $\gamma = 1$ and weak order $\beta = 1$. It has a higher order of convergence in the strong sense than the Euler-Maruyama scheme, but gives no improvement in the weak sense.
- The Milstein scheme does **not correspond to a stochastic Taylor scheme** for an hierarchical set for the week convergence.
- **Pathwise convergence** was already considered for RODEs and is also interesting for SODEs because numerical calculations of the random variables Y_n in the numerical scheme above are carried out **path by path**. (P.E. Kloeden and A. Neuenkirch, The pathwise convergence of approximation schemes for stochastic differential equations, LMS J. Comp. Math. 10 (2007))
- Itô stochastic calculus is, however, an L^2 or a **mean-square calculus** and not a pathwise calculus.
- For the week convergence and their construction especially Hierarchical sets, we suggest the monograph of *Kloeden, Peter E., Platen, Eckhard* **Numerical Solution of Stochastic Differential Equations**.

- More details about stability, consistency and scheme properties
- Implicit schemes and relationship to stability
- Multistep stochastic schemes for SDEs
- Extrapolation (Richardson-like) stochastic schemes for SDEs
- Variance Reduction: Reduction of the number of sample for getting same results
- . . .

Part III

Numerical Schemes for SPDEs

- *Stochastic Method of Lines for solving SPDEs*
- *Domain decomposition method for SPDEs*
- *The Wick-Method for solving SPDEs*

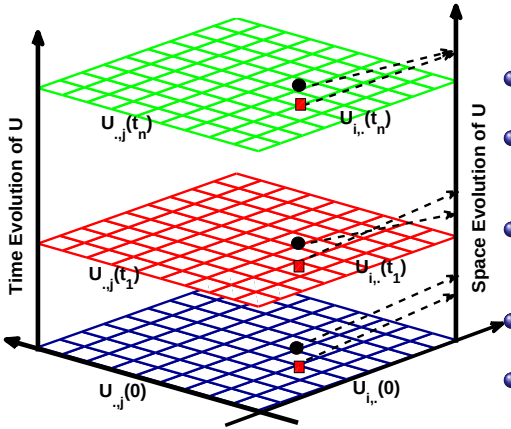
Method of lines for SPDEs

- A. Rößler, M. Seaïd, M. Zahri, **Method of Lines for Stochastic Boundary-Value Problems with Additive Noise** Appl. Math. Comput., Vol. 199, No. 1, 301-314 (2008).
- M. Zahri, **Stochastic diffusion problems: Comparison between Euler-Maruyama and Runge-Kutta schemes**, Jour. Num. Math. Stoch., Vol. 1, pp 65-76, 2009
- A. Rößler, M. Seaïd, M. Zahri, **Numerical Simulation of Stochastic Replicator Models in Catalyzed RNA-like Polymers**, Math and Computers in Simulation Vol. 79, Issue 12, P. 3577-3589 (2009)
- K. Mohamed, M. Seaïd and M. Zahri, **A finite volume method for scalar conservation laws with stochastic time-space dependent flux function**, Journal of Computational and Applied Mathematics, Volume 237, Issue 1, 1 January 2013, Pages 614-632
- M. El-Amrani, M. Seaïd and M. Zahri, **A stabilized finite element method for stochastic incompressible Navier-Stokes equations**, International Journal of Computer Mathematics, Volume 89, 2012 - Issue 18

The algorithm of the MOL for deterministic evolutionary PDE is based on

- replacing the spatial derivatives in the PDE with algebraic approximations: The approximations methods finite difference, finite element, or finite volume methods could be used for the spatial discretization.
- the PDE is no-longer dependent from the spatial derivatives.
- transforming the PDE to time dependent system of ODE.
- solving the time dependent system of ODE by using classical numerical approximations.

The Method of Lines: Illustration



- Discretize the space
- Change the SPDE to a system of linear SDE
- The System SDE is time dependent system! Only
- Integrate the SSDE
- We get one realization of the solution
- repeat the process until the results are 'reasonable'

Given the two **stochastic Advection-diffusion** problems with additive noise of the following form:

$$\begin{aligned} \partial_t u - \nabla(d(\mathbf{x})\nabla u) + a(\mathbf{x})\nabla u &= \sigma(\mathbf{x})\dot{W}(t,\mathbf{x}), \\ u(t,\mathbf{x},\omega) &= 0, \qquad (t,\mathbf{x},\omega) \in [0,T]\times\partial\mathcal{D}\times\Omega, \\ u(0,\mathbf{x},\omega) &= u^0(\mathbf{x},\omega), \qquad (\mathbf{x},\omega) \in \mathcal{D}\times\Omega, \end{aligned} \tag{41}$$

where

- $d(\mathbf{x}) := d(x,y)$ represents the diffusion $a(\mathbf{x}) := a(x,y)$ the advection coefficient function in the x- and the y-directions,
- $W(t,\mathbf{x}) := W(t,x,y) = (W_x(t,x), W_y(t,y))$ is a two dimensional random process representing a forcing term
- $\sigma(x,y)$ is a correction standard deviation.

Using a second order central **finite difference method**, the SPDE (41) could be discretized as ($i = 1, \ldots, n$, $j = 1, \ldots, m$)

$$\begin{aligned} dU_{i,j}(t) &= \Big(d_x \frac{U_{i+1,j}(t) - 2U_{i,j}(t) + U_{i-1,j}(t)}{(\Delta x_i)^2} \\ &+ d_y \frac{U_{i,j+1}(t) - 2U_{i,j}(t) + U_{i,j-1}(t)}{\left(\Delta y_j\right)^2} \\ &- a_x \frac{U_{i+1,j}(t) - U_{i-1,j}(t)}{(\Delta x_i)} \\ &+ a_y \frac{U_{i,j+1}(t) - U_{i,j-1}(t)}{\left(\Delta y_j\right)} \Big) dt + \sigma_i dt dW(t, x_i, y_j) \end{aligned} \tag{42}$$

The system (42) can be written in the following stochastic differential form

$$dU_{i,j}(t) = F(t, U_{i,j}(t))dt + G(t, U_{i,j}(t))dW_t^{i,j}, \tag{43}$$

For numerical solving of the SDEs the class of stochastic Euler-Maruyama and Runge-Kutta schemes could be used.

The s-stage SRK method applied to the SDE (42) is given by

$$\begin{aligned} Y_{n+1} &= Y_n + \sum_{i=1}^{s} \alpha_i \, F(t_n + c_i \, \Delta t, H_i) \, \Delta t + \sum_{k=1}^{M} \sum_{i=1}^{s} \beta_i G^k(t_n) \, \hat{I}_{(k)} \\ Y_0 &= X_{t_0}, \end{aligned} \tag{44}$$

where the SRK stages are defined for $i = 1, \ldots, s$ as

$$H_i = Y_n + \sum_{j=1}^{s} A_{ij} \, F(t_n + c_j \, \Delta t, H_j) \, \Delta t + \sum_{l=1}^{M} \sum_{j=1}^{s} B_{ij} G^l(t_n) \hat{I}_{(l)}. \tag{45}$$

The random variables $\hat{I}_{(k)}$ used by the SRK method are, for example, independent identically $N(0, \Delta t)$ distributed or simply independent identically distributed with probabilities $P\left(\hat{I}_{(k)} = \pm\sqrt{3\Delta t}\right) = \frac{1}{6}$, and $P\left(\hat{I}_{(k)} = 0\right) = \frac{2}{3}$, $\qquad k = 1, \ldots, M$.
(We refer to the works of Andreas Rößler [10, 11]).

The considered error of the **weak SRK approximation** Y is bounded as

$$\left|\left\langle \psi(X)\right\rangle - \left\langle \psi(Y(t))\right\rangle\right| \leqslant C_{\psi}\,(\Delta t)^2,$$

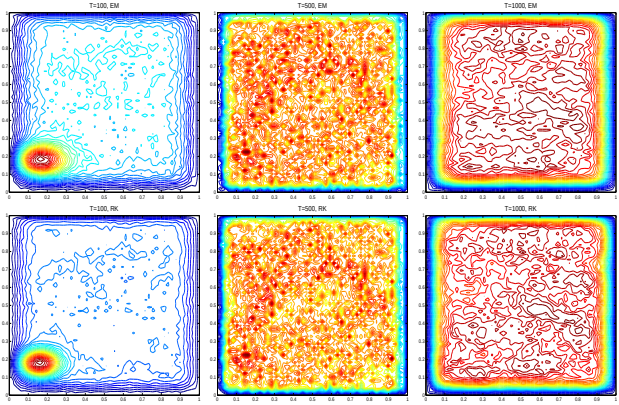
with some constant $C_{\psi} > 0$ for all functionals $\psi \in C_P^6(R^M, R)$ with polynomial growth and sufficient small time step Δt.
The Butcher tableau of the SRK with the number of stages $s = 2$ is

$$\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} & 1 & 0 \end{array} \tag{46}$$

One can generate M of realizations for simulating the solution $Y_N^{(m)}$ for $m = 1, \ldots, M$, and to estimate the expected value of the solution Y_N at final time T one can use:

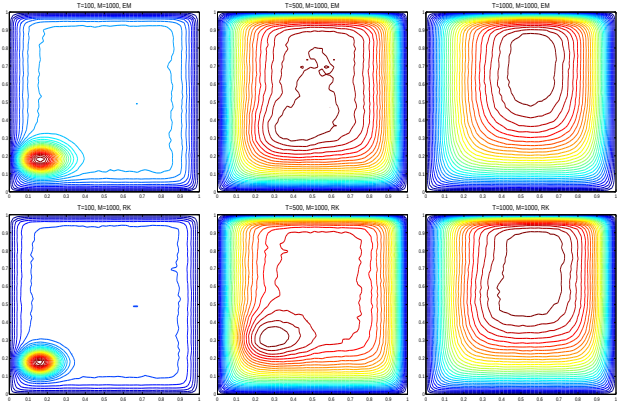
$$\left\langle \psi(Y_N)\right\rangle \approx \frac{1}{M}\sum_{m=1}^{M} \psi\left(Y_N^{(m)}\right).$$

MOL: EM vs. RK (Three Simulation)



- Contour of the solution using EM (first row) and RK schemes (second row),
- **Similar behavior of the solutions EM vs RK**

MOL: EM vs. RK (Mean Solution)



Contour of the mean solution of EM-Simulations and RK-Simulations (second row). ($T = 100, 500, 1000$ represents the number of time steps, $M = 1000$ refers the number of realization)

- Contour of the mean solution using EM (first row) and RK schemes (second row),
- **Different behavior of the solutions EM vs RK. EM is more diffusive than RK**
- The diffusibility is an important factor for stability of the solution.

Numerical solution of stochastic partial differential equations (SPDEs) with additive noise on overlapping subdomains

- M. Zahri, **Barycentric interpolation of interface-solution for solving stochastic partial differential equations on non-overlapping subdomains with additive multi-noises**, International Journal of Computer Mathematics 2018, VOL. 95, NO. 4, 645-685
- M. Zahri, **Numerical solution of stochastic partial differential systems with additive noise on overlapping subdomains**, 7th International Conference on Approximation Methods and Numerical Modelling in Environment and Natural Resources (MAMERN17 VII-2017, Oujda, Morocco, 17-20 May 2017), B. Amaziane et al. (Eds.), pp. 198-201.
- M. Zahri, **Interface interpolation of the parallel Schwarz method for solving Dirichlet IBV systems**, (under review)

- Alternating Schwarz method (Schwarz 1870, ASM) and Parallel Schwarz method, (Lions 1988, PSM) for solving deterministic PDEs!



(A) Non-overlapped dom. (B) Overlapped dom.

ABBILDUNG: Non-overlapping noises (left), Overlapping noises (right)

- ASM: The solution in D_1 is a Boundary condition of the Sol. in D_2 and vice versa $\Rightarrow$ weak for stochastic partial differential equations (SPDEs).
- PSM: The noise separation of the non-overlapped subdomains leads to the construction of higher order method for approximating SPDEs.
- PSM: The changes in the overlapped region is not an alternating process like in the deterministic case? How to deal with the new function!

Numerical solution of SPDEs with additive noise on non-overlapping subdomains

PART I

1. *Modeling of SPDS on non-overlapping subdomains*
2. *The equivalent problem*
3. *Barycentric interpolation method (BIM)*
4. *DDM-MOL-BIM descritization of SPDS*
5. *Itô-Taylor family schemes*
6. *Numerical experiments*

More details about the result of this work could be found in:

- M. Zahri, Barycentric interpolation of interface-solution for solving stochastic partial differential equations on non-overlapping subdomains with additive multi-noises, International Jour. of Comp. Math., March 2018. VOL. 95, NO. 4, 645 – 685

Initial Problem

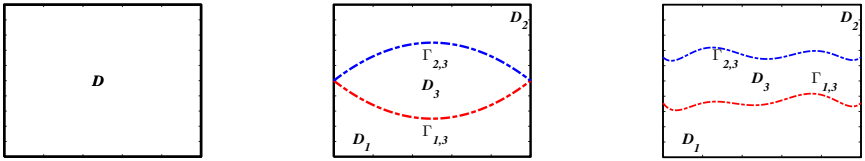
Assume that $\mathbf{u}$ is the exact solution of the following stochastic advection-diffusion problem with additive multi-noise:

$$\begin{aligned} \mathbf{u}_t &= \nabla \cdot (\mathbf{d}\nabla\mathbf{u}) + \mathbf{a} \cdot \nabla\mathbf{u} + \sum_{k=1}^{m} \sigma^k(t, \mathbf{x}, \omega)\eta^k(t, \mathbf{x}, \omega), \qquad (47) \\ & \mathbf{u}(t, \mathbf{x}, \omega) = 0, \quad (t, \mathbf{x}, \omega) \in [0, T] \times \partial D \times \Omega, \\ & \mathbf{u}(0, \mathbf{x}, \omega) = \mathbf{u}^0(\mathbf{x}, \omega), \qquad (\mathbf{x}, \omega) \in D \times \Omega, \end{aligned}$$

where the domain D is a closed subset from $\mathbb{R}^2$ with smooth boundary, the input matrix $\mathbf{d} = \mathbf{d}(\mathbf{x}) := d(x, y)$ represents the diffusion matrix function in different directions, the coefficient $\mathbf{a} = \mathbf{a}(\mathbf{x}) := a(x, y)$ is the velocity field in the x- and the y-directions, $\eta^k(t, \mathbf{x})$ is an time-space white noise process representing a forcing term, which also depends on the spatial coordinate, $\sigma^k(t, \mathbf{x})$ is a time-space correction standard deviation for $k = 1, \ldots, m$.

Domain decompositions (Three subdomains)

- Two domain decomposition of the domain D
- $\Gamma_{1,3}$ and $\Gamma_{2,3}$ are interfaces of the subdomains D_s, $s = 1, 2, 3$
- Solutions of SPDE on D and on $D_1 \cup D_2 \cup D_3$ are equivalent !!!



(A) Domain D (B) Eye-Cut (C) Colon-Cut

ABBILDUNG: Type of horizontal nonlinear interfaces.

Equivalent Problem in case of two noises

- Non-overlapping subdomains $\mathcal{D}_s \subset \mathbb{R}^2$: $\mathcal{D}_s \cap \mathcal{D}_r = \Gamma_{s,r}$, $s \neq r$
- Stochastic process $\boldsymbol{U}^s : [0,T] \times \mathcal{D}_s \times \Omega \longrightarrow \mathbb{R}$, $s, r = 1, 2, 3$
- $(\boldsymbol{U}^1, \boldsymbol{U}^2, \boldsymbol{U}^3)$ the exact solution of the linear SPDS:

$$\begin{cases} \boldsymbol{U}_t^1 = \nabla \cdot (d_1 \nabla \boldsymbol{U}^1) - a_1 \cdot \nabla \boldsymbol{U}^1 + \sigma^1 \eta^1, & \text{on} \quad \mathcal{D}_1, \\ \boldsymbol{U}_t^2 = \nabla \cdot (d_2 \nabla \boldsymbol{U}^2) - a_2 \cdot \nabla \boldsymbol{U}^2 + \sigma^2 \eta^2, & \text{on} \quad \mathcal{D}_2, \\ \boldsymbol{U}_t^3 = \nabla \cdot (d_3 \nabla \boldsymbol{U}^3) - a_3 \cdot \nabla \boldsymbol{U}^3 + \sigma^3 \eta^3, & \text{on} \quad \mathcal{D}_3, \end{cases} \tag{48}$$

where for $s = 1, 2, 3$, η^s, are non-necessary correlated Gaussian white noises and a_s, d_s are advection and diffusion coefficients, σ^s standard deviation.

In the equivalent Problem

- $\Gamma_{1,3}$ and $\Gamma_{2,3}$ are boundaries of the subdomains D_s, $s = 1, 2, 3$
- How to approximate $\boldsymbol{U}^s$ on $\Gamma_{1,3}$ and $\Gamma_{2,3}$ for $s = 1, 2, 3$.

The DD problem is transformed into a system of the form:

$$\begin{cases} \boldsymbol{U}_t^1 &= d_1 \Delta \boldsymbol{U}^1 - a_1 \nabla \boldsymbol{U}^1 + \sigma^1 \eta^1 & \text{on} \quad \mathcal{D}_1, \\ \boldsymbol{U}_t^2 &= d_2 \Delta \boldsymbol{U}^2 - a_2 \nabla \boldsymbol{U}^2 + \sigma^2 \eta^2 & \text{on} \quad \mathcal{D}_2, \\ \boldsymbol{U}_t^3 &= d_3 \Delta \boldsymbol{U}^3 - a_3 \nabla \boldsymbol{U}^3 + \sigma^3 \eta^3 & \text{on} \quad \mathcal{D}_3, \end{cases} \tag{49}$$

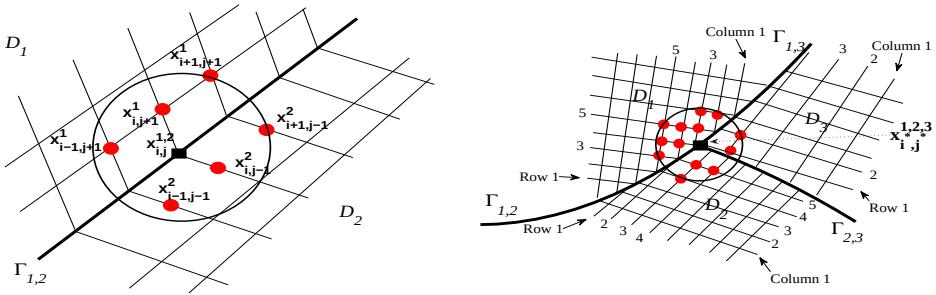
where the interface solution satisfies

$$\mathbf{w}^{1,3} = \boldsymbol{U}_t^3 = \boldsymbol{U}_t^1 \quad \text{on} \quad \Gamma_{1,3} \quad \mathbf{w}^{2,3} = \boldsymbol{U}_t^3 = \boldsymbol{U}_t^2 \text{ on} \quad \Gamma_{2,3}$$

- How to approximate $\mathbf{w}^{1,3}$ and $\mathbf{w}^{2,3}$? to obtain
- Equivalent Problem: Solving $\boldsymbol{U}$ on $\mathcal{D}$ (initial problem)
- such that $\boldsymbol{U}^s = \boldsymbol{U} \mid_{\mathcal{D}_s \setminus \partial \mathcal{D}_s}$ for $s = 1, 2, 3$ and $\boldsymbol{U}^s = \mathbf{w} \mid_{\partial \Gamma_{1,2}}$.

Comparing the initial prob. with DD one is an interface-problem!

Interpolating the interface solution of SPDEs



(A) Point on Linear-interface **(B)** Point on Non-linear-interface

Abbildung: Graphical illustration of the BIM method.

Which neighbors grid points are candidates for the BIM method !

For $m \in \mathbb{N}$, define the **neighbors grid-points** index set $N_{i,j}^m$ as

$$N_{i,j}^m := \left\{(i+a, j+b) \middle| |a| = |b| = m; a, b \in \mathbb{Z}\right\}, \tag{50}$$

The **coefficients set** $\alpha_{i^*,j^*} \in C_{i,j} \subset]0,1[$, are

$$\begin{aligned} C_{i,j} := \Bigg\{ & \left(\frac{1-2\alpha}{4}\right)_{i+2,j-2}, \left(\frac{1-2\alpha}{4}\right)_{i+2,j+2}, \left(\frac{\alpha}{2}\right)_{i+1,j-1}, \left(\frac{\alpha}{2}\right)_{i+1,j+1}, \\ & \left(\frac{1-2\alpha}{4}\right)_{i-2,j-2}, \left(\frac{1-2\alpha}{4}\right)_{i-2,j+2}, \left(\frac{\alpha}{2}\beta^+\right)_{i-1,j-1}, \left(\frac{\alpha}{2}\right)_{i-1,j+1} \Bigg\}. \end{aligned}$$

and satisfies the following convex combination

$$\boxed{\sum_{\alpha_{i^*,j^*} \in C_{i,j}} \alpha_{i^*,j^*} = \sum_{(i^*,j^*) \in N_{i,j}^1 \cup N_{i,j}^2} \alpha_{i^*,j^*} = 1.} \tag{51}$$

Under appropriate conditions on IS $\boldsymbol{U}^0$ in $L^2(\mathcal{D})$, there exist a unique solution $\boldsymbol{U}^{DD}$ in $C(\mathbb{T}, L^2(\mathcal{D})) \cap L^2(0,T; H_0^1(\mathcal{D}))$, such that

$$\boldsymbol{U}^{DD}(\mathbf{x},t) \;:=\; \mathbf{w}(\mathbf{x},t)|_{\partial\Gamma_{1,2}} + \sum_{s=1}^{3} \boldsymbol{U}^s(\mathbf{x},t)|_{\mathcal{D}_s \setminus \partial\mathcal{D}_s}, \tag{52}$$

where $\boldsymbol{U}^s$ is the solution of the sub-problem s for $s = 1, 2, 3$.
The non-overlapping interface solution is

$$\mathbf{w}(\mathbf{x}_{i,j},0) \;=\; \boldsymbol{U}^{DD}(\mathbf{x}_{i,j},0) = \boldsymbol{U}^s(\mathbf{x}_{i,j},0),$$

and for $t > 0$, we have

$$\boxed{\mathbf{w}(\mathbf{x}_{i,j},t) = \sum_{(i^*,j^*)\in N_{i,j}^1 \cup N_{i,j}^2} \alpha_{i^*,j^*} \boldsymbol{U}^{DD}(\mathbf{x}_{i^*,j^*}, t-\Delta t),} \tag{53}$$

The initial and Dirichlet boundary conditions are

$$\begin{array}{lll} \boldsymbol{U}^s(\mathbf{x},0) & = \boldsymbol{U}(\mathbf{x},0)|_{\mathcal{D}_s} & \text{for} \quad s = 1,2,3. \\ \boldsymbol{U}^s(\mathbf{x},t) & = 0 & \text{for} \quad (\mathbf{x},t) \in \partial\mathcal{D} \times \mathbb{T}. \end{array} \tag{54}$$

- Finite differences and the: forward in time centred in space (FTCS),
- Constant temporal step Δt. Ansatz $\Delta x = \Delta y$.
- The method of lines (MOL): System (49) $\Rightarrow$ Semi-linear system of systems:

$$\begin{aligned} dU^s_{i,j}(t) &= \Big[(-2\alpha)U^s_{i,j}(t) + \frac{\alpha}{2}\boldsymbol{U}^s_{i-1,j}(t) + \frac{\alpha}{2}\beta^+ U^s_{i+1,j}(t) \\ &\quad + \frac{\alpha}{2}\boldsymbol{U}^s_{i,j-1}(t) + \frac{\alpha}{2}\beta^+ U^s_{i,j+1}\Big]dt + \mathbf{G}^s(t)d\mathbf{W}^s_t \end{aligned} \tag{55}$$

where $s = 1, 2, 3$ and

- $\mathbf{G}^s(t)$ Diffusion matrix, $d\mathbf{W}^s$ Brownian motion vector,
- $\alpha = \dfrac{2d\Delta t}{(\Delta x)^2}$, $= 1 - \dfrac{a}{2d}\Delta x$, and $\beta^+ = 1 + \dfrac{a}{2d}\Delta x$.

The resulting system (55) is a time-problem!

In the framework of numerical stochastics: From (55) we obtain

$$\begin{cases} d\boldsymbol{U}^1(t) &= \mathbf{K}^1(t,\boldsymbol{U}^1(t))dt + \mathbf{G}^1(t)d\mathbf{W}^1(t) \quad \text{on} \quad \mathcal{D}_1 \setminus \Gamma_{1,2}, \\ d\boldsymbol{U}^2(t) &= \mathbf{K}^2(t,\boldsymbol{U}^2(t))dt + \mathbf{G}^2(t)d\mathbf{W}^2(t) \quad \text{on} \quad \mathcal{D}_2 \setminus \Gamma_{1,2}, \\ d\boldsymbol{U}^3(t) &= \mathbf{K}^3(t,\boldsymbol{U}^3(t))dt + \mathbf{G}^3(t)d\mathbf{W}^3(t) \quad \text{on} \quad \Gamma_{1,2} \\ \mathbf{w}^{1,3} &= \boldsymbol{U}_t^3 = \boldsymbol{U}_t^1 \quad \text{on} \quad \Gamma_d \quad \mathbf{w}^{2,3} = \boldsymbol{U}_t^3 = \boldsymbol{U}_t^2 \text{ on} \quad \Gamma_u \end{cases}$$

Which is equivalent to

$$d\boldsymbol{U}(t) = \mathbf{K}(t,\boldsymbol{U}(t))dt + \mathbf{G}(t)d\mathbf{W}(t) \quad \text{on} \quad \mathcal{D},$$

where

- $\mathbf{K}^s(t,\boldsymbol{U}^s(t))$ is the drift vector, for $s = 1,2,3, \Rightarrow \mathbf{K}^s = \mathbf{K}\mid_{\mathcal{D}_s \setminus \partial\mathcal{D}_s}$.
- $\mathbf{G}^s(t)$ is the diffusion matrix, for $s = 1,2,3, \Rightarrow \mathbf{G}^s = \mathbf{G}\mid_{\mathcal{D}_s \setminus \partial\mathcal{D}_s}$.

For $s = 1, 2, 3$ and $U_0^s = \xi_0$, and U_n^s approximation of $\boldsymbol{U}^s(\cdot, t_n)$ for a given time discretization $\Delta t = t_{n+1} - t_n$. The Itô-Taylor for the system (49) is...

$$U_{n+1}^s = Y_n^s + I_{(0)}[\mathbf{K}_s^i(t, U_n^s)]_{\Delta t_n} + \sum_{j \in F_m^1} I_{(j)}[\mathbf{G}_j^s(t, U_n^s)]_{\Delta t_n} \tag{56}$$

$$+ \sum_{(j,j) \in F_m^2} I_{(j,j)}[L^j \mathbf{G}_j^s(t, U_n^s)]_{\Delta t_n} \tag{57}$$

$$+ \sum_{(j,0) \in F_m^1 \times \{0\}} I_{(j,0)}[L^j \mathbf{K}_i^s(t, U_n^s)]_{\Delta t_n}$$

$$+ \sum_{(0,j) \in \{0\} \times F_m^1} I_{(0,j)}[L^0 \mathbf{G}_j^s(t, U_n^s)]_{\Delta t_n}$$

$$+ I_{(0,0)}[L^0 \mathbf{K}_i^s(t, U_n^s)]_{\Delta t_n} \tag{58}$$

- Euler Maruyama scheme of order 0.5 $\Rightarrow$ (56),
- Milstein (or Taylor order one) scheme $\Rightarrow$ (56)-(57),

$$
\begin{aligned}
Y_{n+1}^{s} \;=\; & \ldots + \sum_{(j,j,j)\in F_m^3} I_{(j,j,j)}[L^j L^j \mathbf{G}_j^s(t,U_n^s)]_{\Delta t_n} \\
& + \sum_{(j,j,0)\in F_m^2\times\{0\}} I_{(j,j,0)}[L^j L^j \mathbf{K}_i^s(t,U_n^s)]_{\Delta t_n} \\
& + \sum_{(j,0,j)\in F_m\times\{0\}\times F_m} I_{(j,0,j)}[L^j L^0 \mathbf{G}_j^s(t,U_n^s)]_{\Delta t_n} \\
& + \sum_{(0,j,j)\in\{0\}\times F_m^2} I_{(0,j,j)}[L^0 L^j \mathbf{G}_j^s(t,U_n^s)]_{\Delta t_n} \\
& + \sum_{(j,j,j,j)\in F_m^4} I_{(j,j,j,j)}[L^j L^j L^j \mathbf{G}_j^s(t,U_n^s)]_{\Delta t_n}, \qquad (59)
\end{aligned}
$$

- Taylor scheme of order 2.0 $\Rightarrow$ (56)-(59)
- $L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} a_t^k \frac{\partial}{\partial x^k} + \frac{1}{2}\sum_{k,i=1}^{d}\sum_{j=1}^{m} b^{i,j} b^{k,j} \frac{\partial^2}{\partial x^i \partial x^k}$,

- The process Y_n is a **strong approximation** of $\boldsymbol{U}$ with order $\gamma = 0.5, 1, 1.5, \ldots$, if it exists a constant $K_{p,T} > 0$ such that

$$E_s^{\gamma}(Y_n) := \left(E\,|Y_T^{\gamma} - \boldsymbol{U}_T|^p\right)^{\frac{1}{p}} \leqslant K_{p,T}(\Delta t)^{\gamma}, \tag{60}$$

- The process Y_n is a **weak approximation** of $\boldsymbol{U}$ with order $\beta = 1, 2, 3, \ldots$, if it exists a constant $K_{g,T} > 0$ such that

$$E_w^{\beta}(Y_n) := \left|E\,g\left(Y_T^{\beta}\right) - E\,g(\boldsymbol{U}_T)\right| \leqslant K_{g,T}(\Delta t)^{\beta}, \tag{61}$$

  where g any polynomial function, Δt fixed time step and $p = 1$ od 2.

- (60)-(61) $\Rightarrow \lim_{\Delta t \to 0} E_s^{\gamma}(Y_n) = \lim_{\Delta t \to 0} E_w^{\beta}(Y_n) = 0.$

- The numerical approximations above are given in $\mathbb{L}^2$.

For the domain Eye-cut, we start by the following initial solution:

$$\mathbf{u}(0,x,y) = \frac{4}{3}\sum_{k=1}^{3} \exp\Big(- \lambda_k \big((x - x_{c_k})^2 + (y - y_{c_k})^2\big)\Big), \tag{62}$$

where $\lambda_k = 15 \cdot 10^{-6}$ and

$$(x_{c_k}, y_{c_k}) = \begin{cases} (x_{50}, y_{10}) = (0.5, 0.1) & \text{if} \quad k = 1, \\ (x_{50}, y_{50}) = (0.5, 0.5) & \text{if} \quad k = 2, \\ (x_{50}, y_{90}) = (0.5, 0.9) & \text{if} \quad k = 3. \end{cases}$$

and for the domain Colon-Cut, we have

$$(x_{c_k}, y_{c_k}) = \begin{cases} (x_{90}, y_{10}) = (0.9, 0.1) & \text{if} \quad k = 1, \\ (x_{10}, y_{50}) = (0.1, 0.5) & \text{if} \quad k = 2, \\ (x_{90}, y_{90}) = (0.9, 0.9) & \text{if} \quad k = 3. \end{cases}$$

We compare the reference solution $\mathbf{u}_t^R$ with the mean solution $\mathbf{u}_t^M$, given by

$$\mathbf{u}^M(t,\mathbf{x}) = \frac{1}{1000}\sum_{v=1}^{M} \mathbf{u}^{\Sigma}(t,\mathbf{x},\omega_v), \qquad \Sigma \quad \text{interface type} \tag{63}$$

Example 1. Convection diffusion on a Eye-like domain:

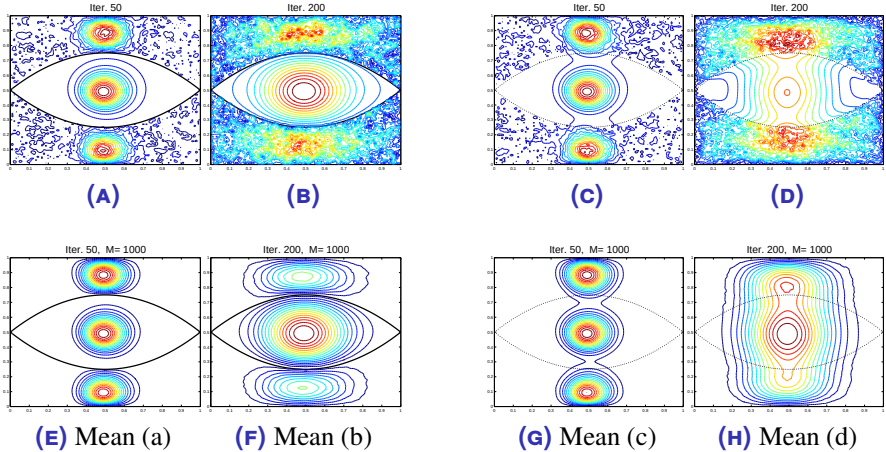


ABBILDUNG: Sample solution of SPDEs: (left block) non-interpolated interface, (right block) interpolated interface

Example 2. Convection diffusion on a Colon-like domain:

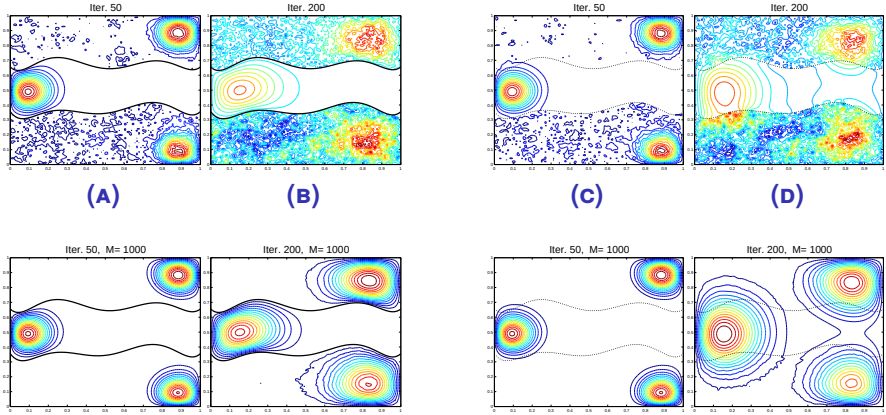


ABBILDUNG: Sample solution of SPDEs: (left block) non-interpolated interface, (right block) interpolated interface

Concluding remarks

- Construction of SPDS model on overlapping subdomains.
- Derivation of BIM from finite differences.
- Combination of DDM-MOL-BIM.
- Solving deterministic and stochastic overlapped problems.

Outlook

- Solving SPDEs on overlapping subdomains using real parameters ...
- Applying the BIM using other schemes: Crank Nicolson, Relaxation ...
- Applying the BIM using other discretization methods: finite volume, finite elements and meshless ...

Wick-Method for SPDEs

- H. Manouzi, M. Seaïd, M. Zahri, **Wick-Stochastic Finite Element Solution of Reaction-Diffusion Problems, J. Comp. Applied Math. Vol 203, pp:516-532 (2007)**
- **M. Zahri, M. Seaïd, H. Manouzi, M. El-Amrani, Wiener-Itô Chaos Expansions and Finite-Volume Solution of Stochastic Advection Equations, Finite Volume and Application IV: pp: 525-538, (2005)**

Multi-Indices Set $\mathcal{I}$

We denote by $\mathcal{I}$ the set of all multi-indices $(\alpha_1, \alpha_2, \dots) \in \mathbb{N}_0^{\mathbb{N}}$ with only finitely many $\alpha_i \neq 0$,such that

- $\ell(\alpha) := \max\{i : \ \alpha_i \neq 0\}$,
- $\alpha + \beta := (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots)$,
- $\alpha! := \alpha_1!\alpha_2! \dots$, $|\alpha| := \sum_i \alpha_i$,
- $\alpha \prec \beta$ if $\alpha_i < \beta_i$ for all $i \in \mathbb{N}$

Example

For $\alpha = (1, 2, 0, 1, 0, \dots)$, $\beta = (2, 1, 0, \dots) \in \mathcal{I}$, we have

- $\ell(\alpha) = 4$ and $\ell(\beta) = 2$
- $\alpha + \beta := (3, 3, 0, 1, \dots)$,
- $\alpha! := 1!2!0!1! = 2 \dots$, $|\alpha| = 4$, $\beta! := 2!1! = 2 \dots$, $|\beta| = 2$,
- $\alpha \prec \beta$.

For $n \in \mathbb{N}_0$, $x \in \mathbb{R}$ define the Hermite polynomial

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} \left(e^{-\frac{x^2}{2}} \right),$$

and for $n \in \mathbb{N}$ we define the Hermite functions

$$\xi_n = \pi^{-\frac{1}{4}} \left((n-1)! \right)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{n-1}(\sqrt{2}x).$$

Define the tensor-product

$$\xi_\delta(\mathbf{x}) := \xi_{\delta_1} \otimes \xi_{\delta_2} \otimes \cdots \otimes \xi_{\delta_d}(\mathbf{x}) = \xi_{\delta_1}(x_1) \otimes \xi_{\delta_2}(x_2) \otimes \cdots \otimes \xi_{\delta_d}(x_d).$$

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decaying $C^\infty(\mathbb{R}^d)$ functions, $\mathcal{S}'(\mathbb{R}^d)$ be its dual space of tempered distributions, $(\mathcal{S}(\mathbb{R}^d), \mathfrak{B}, \mu)$ a probability space .For $\alpha \in \mathcal{I}$, define the random variable

$$H_\alpha(\omega) = \prod_{j=1}^{\ell(\alpha)} h_{\alpha_j}\left(< \omega, \xi_{\delta_j} >\right), \qquad \omega \in \mathcal{S}'(\mathbb{R}^d),$$

where δ_j is some ordering of d-dimensional multi-indices such that

$$i < j \Longrightarrow |\delta_i| < |\delta_j|.$$

The family $\{H_\alpha\}_{\alpha\in\mathcal{I}}$ constitutes an **orthogonal basis** for $L^2(\mu)$ (see [3]):

$$\boldsymbol{E}[H_\alpha H_\beta] = \delta_{\alpha\beta}\alpha!.$$

Thus, any function $f \in L^2(\Omega)$ has a unique representation

$$f = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha, \qquad \| f \|^2_{L^2(\Omega)} := \int_\Omega f(\omega)^2 d\mu(\omega) = \sum_{\alpha\in\mathcal{I}} f_\alpha^2 \alpha!.$$

This representation is called **Wiener-Itô chaos expansion**.

Here $f_\alpha \in \mathbb{R}$ denotes the αth chaos coefficient of function f. Let V denotes any real separable Hilbert space with inner product $(\cdot,\cdot)_V$ and let $k \in \mathbb{R}$ and $\rho \in [0,1]$. Define

$$^{\rho,k,V} := \left\{ f = \sum_{\alpha \in \mathcal{I}} f_\alpha H_\alpha : \quad f_\alpha \in V,\ \forall\, \alpha \in \mathcal{I} \text{ and } \| f \|_{\rho,k,V} < \infty \right\},$$

where the norm $\| \cdot \|_{\rho,k,V}$ is induced by the inner product

$$(f,g)_{\rho,k,V} := \sum_{\alpha \in \mathcal{I}} (f_\alpha, g_\alpha)(\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha},\ \ f,g \in {}^{\rho,k,V}, \tag{64}$$

where $(2\mathbb{N})^{k\alpha} = \prod_j (2j)^{k\alpha_j}$. Here $^{\rho,k,V}$ is called the Kondratiev space of test functions and its dual is denoted $^{-\rho,-k,V'}$, and is the Kondratiev space of distributions [4, 1].

Given $f = \sum_{\alpha \in \mathcal{I}} f_\alpha H_\alpha$, $g = \sum_{\beta \in \mathcal{I}} g_\beta H_\beta \in L^2(\mu)$, the Wick product is defined by

$$\boxed{f \diamond g := \sum_{\alpha,\beta \in \mathcal{I}} f_\alpha g_\beta H_{\alpha+\beta}.} \tag{65}$$

The Wick product satisfies:

- The commutative, associative and distributive laws.
- $\left(^{\rho,k,V}, \diamond\right)$ and $\left(^{-\rho,-k,V'}, \diamond\right)$ are algebras. (see [7].)
- If $f = \sum_{\alpha \in \mathcal{I}} f_\alpha H_\alpha$, then $f_0 := \boldsymbol{E}[f]$ (first chaos coefficient).
- For any $f, g \in L^2(\mu)$, we have $\boldsymbol{E}[f \diamond g] = \boldsymbol{E}[f]\boldsymbol{E}[g]$.
  (This property does not hold for the ordinary product.)

Given the stochastic reaction-diffusion equations

$$\begin{aligned} \partial_t u - \operatorname{div}\Big(D(\mathbf{x},\omega) \diamond \nabla u\Big) &= f(t,\mathbf{x},\omega), && (t,\mathbf{x},\omega) \in (0,T) \times \mathcal{D} \times \Omega, \\ u(t,\mathbf{x},\omega) &= 0, && (t,\mathbf{x},\omega) \in [0,T] \times \partial\mathcal{D} \times \Omega, \\ u(0,\mathbf{x},\omega) &= u^0(\mathbf{x},\omega), && (\mathbf{x},\omega) \in \mathcal{D} \times \Omega, \end{aligned} \tag{66}$$

where

- $\mathcal{D} \subset \mathbb{R}^d$ ($d = 1, 2$ or 3) with a smooth boundary $\partial\mathcal{D}$,
- $u(t,\mathbf{x},\omega)$ is, for example, the stochastic concentration of some chemical species,
- $D(\mathbf{x},\omega)$ is the stochastic diffusion coefficient,
- $f(t,\mathbf{x},\omega)$ is the stochastic reaction term,
- $u^0(\mathbf{x},\omega)$ is a given initial data,
- $[0,T]$ represents the time interval.

We define the variational (weak) formulation of equations (66) as follows:
Find $u \in W(0,T;V)$ such that

$$\boxed{\begin{array}{c}(\partial_t u, v) + \mathcal{A}(u,v) = (f,v), \ \forall\ v \in V, \\ u(0) = u^0.\end{array}} \tag{67}$$

where

- X and Y are Hilbert spaces such that $Y \hookrightarrow X \hookrightarrow Y'$, (dense injection)
- $W(0,T;Y) := \left\{ f \in L^2(0,T;Y) :\ \partial_t f \in L^2(0,T;Y) \right\}$ forms a Hilbert space w.r ...
- $(f,g)_{W(0,T;Y)} := \int_0^T (f(s), g(s))_Y \, ds + \int_0^T (\partial_t u(s), \partial_t v(s))_{Y'} \, ds$ is inner product
- $W(0,T;Y) \hookrightarrow C^0(0,T;X)$, such that $C^0(0,T;X)$ is the space of continuous functions from $[0,T]$ into X.
- $\mathcal{A}(u,v) := (D \diamond \nabla u, \nabla v), \qquad \forall\ u, v \in V$ is a bilinear form

the relation between the weak formulation (67) and the corresponding system of deterministic variational problems for the chaos coefficients u_γ with $\gamma \in \mathcal{I}$.
Let $D = \sum_{\gamma \in \mathcal{I}} D_\gamma H_\gamma$, and $u^0 = \sum_{\gamma \in \mathcal{I}} u_\gamma^0 H_\gamma$. Moreover,for all $f, g \in H_0^1(\mathcal{D})$ define the bilinear form $\mathcal{B}_\beta(f, g)$ as $\mathcal{B}_\beta(f, g) = \left(D_\beta \nabla f, \nabla g\right)_0$.

THEOREM

Let $u = \sum_{\gamma \in \mathcal{I}} u_\gamma H_\gamma$ be the solution of the variational problem (67). Then the chaos coefficients u_γ of u satisfy the following system of variational problems:

i) *If $\gamma = 0$, find $u_\gamma \in H_0^1(\mathcal{D})$ such that $u_\gamma(0) = u_\gamma^0$ and*

$$\left(\partial_t u_\gamma, w\right) + \mathcal{B}_0\left(u_\gamma, w\right) = \left(f_\gamma, w\right), \qquad \forall\, w \in H_0^1(\mathcal{D}), \tag{68}$$

ii) *If $\gamma > 0$, find $u_\gamma \in H_0^1(\mathcal{D})$ such that $u_\gamma(0) = u_\gamma^0$ and*

$$\left(\partial_t u_\gamma, w\right) + \mathcal{B}_0\left(u_\gamma, w\right) = \left(f_\gamma, w\right) - \sum_{\alpha < \gamma} \mathcal{B}_{\gamma - \alpha}\left(u_\alpha, w\right), \qquad \forall\, w \in H_0^1(\mathcal{D}), \tag{69}$$

For the proof and the construction, we refer to [8, 9, 5].

Ordered Cutting Set $\mathcal{I}_{N,K}$

Sine the variational problems solved with corresponding to multi-indices set α such that $\alpha \prec \gamma$, For $N, K \in \mathbb{N}$ we define the cutting $\mathcal{I}_{N,K} \subset \mathcal{I}$ as

$$\mathcal{I}_{N,K} = \{0\} \cup \bigcup_{n=1}^{N} \bigcup_{k=1}^{K} \left\{ \alpha \in \mathbb{N}_0^k : \ |\alpha| = n \text{ and } \alpha_k \neq 0 \right\}. \tag{70}$$

The resulting space is given by

$$\overset{-1,k,1}{0} := \left\{ f = \sum_{\alpha \in \mathcal{I}_{N,K}} f_\alpha H_\alpha : \ f_\alpha \in H_0^1(\mathcal{D}), \ \| f \|_{-1,k,1} < \infty \right\}.$$

- for all $\alpha, \gamma \in \mathcal{I}_{N,K}$ either $\alpha \prec \gamma$ or $\gamma \prec \alpha$,
- Such ordering allows us to solve (66) as a sequence of problems.
- The set $\mathcal{I}_{N,K}$ contains $\frac{(N+K)!}{N!K!}$ multi-indices,
- $\overset{-1,k,1}{0}$ is a subspace of V for any choice of N and K.

Step 1. For $n = 0$ do:

(A) Construct the ordered cutting $\mathcal{I}_{N,K}$ as in (70) and set $\gamma = (0, \ldots, 0)$

(B) Compute the vectors $\mathbf{U}_0^n$, $\mathbf{U}_0^{n+1}$ and $\mathbf{F}_0^{n+1}$

Step 2. While $\gamma \in \mathcal{I}_{N,K}$ do: (Sol. of PDE for $\mathcal{I}_{N,K}$)

(C) Locate the subset $\mathcal{I}_\gamma = \left\{\alpha \in \mathcal{I}_{N,K} : \quad \alpha < \gamma\right\}$

(D) Compute the sum $\mathbf{b}_\gamma = \sum_{\alpha<\gamma} \mathbf{S}_{\gamma-\alpha}\mathbf{U}_\alpha^{n+1}$

(E) Update the right hand side $\mathbf{b}_\gamma = \mathbf{M}_\gamma \mathbf{U}_\gamma^n + \Delta t \left(\mathbf{F}_\gamma^{n+1} - \mathbf{b}_\gamma\right)$

(F) Solve for $\mathbf{U}_\gamma^{n+1}$ the linear system: $\left(\mathbf{M}_\gamma + \Delta t \mathbf{S}_\gamma\right)\mathbf{U}_\gamma^{n+1} = \mathbf{b}_\gamma$

(G) Find the next multi-index γ and go to **Step 2**

Step 3. If $n\Delta t \geqslant T$ do (Construction of the Sol. for γ)

(H) Generate normally distributed random numbers $\left\{X_i,\ i = 1, \ldots, R^2\right\}$

(E) For each $r = 1, \ldots, R$, $X^{(r)} = [X_{(r-1)K+j}],\ j = 1, \ldots, R$ form realizations of the solution $u^{(r)}(t_{n+1}, \mathbf{x}_h) = \sum_{\alpha\in\mathcal{I}_{N,K}} U_\alpha^{n+1}(\mathbf{x}_h) H_\alpha\left(X^{(r)}\right)$

(I) Perform some statistics and display the results, go to step 1, set $n = n + 1$

The following test examples are selected:

Test 1: $D = e^{\diamond \dot{W}(\mathbf{x})}$ and $f = 0.$

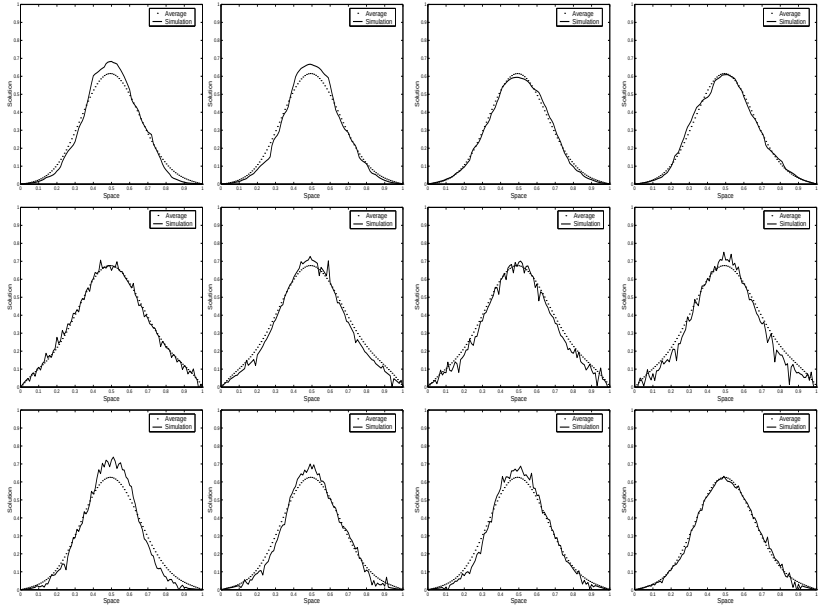
Test 2: $D = 0.1$ and $f = 1 + \dot{W}(\mathbf{x}).$

Test 3: $D = e^{\diamond \dot{W}(\mathbf{x})}$ and $f = 1 + \dot{W}(\mathbf{x}).$

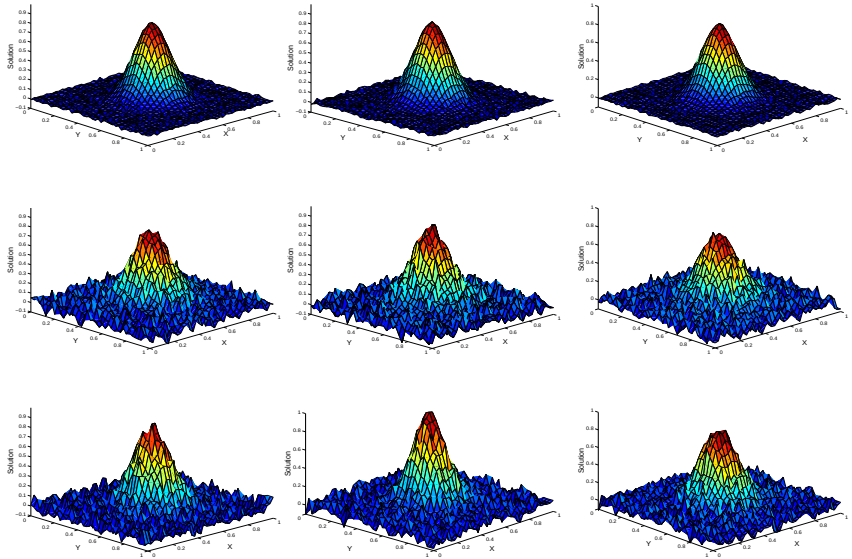
Where

- $\dot{W}(\mathbf{x})$ denotes the white noise,
- the time stepsize Δt is fixed to 10^{-4},
- the initial condition is taken as a normal distribution centered in the spatial domain $\mathcal{D}$ and with variance equal to 0.01.

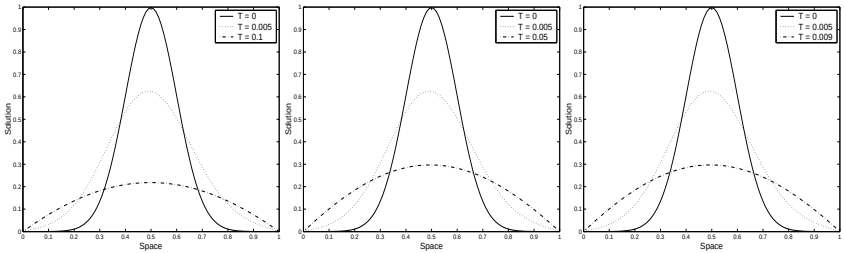
Numerical Simulations one-dim

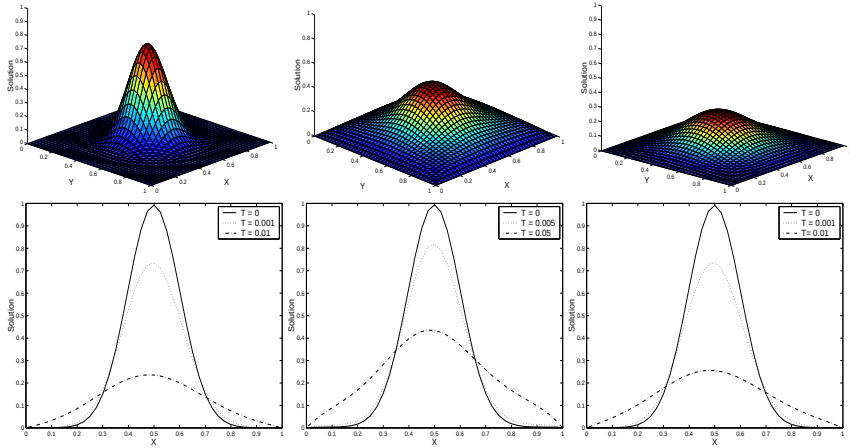


One dimensional tests (TEST1: first row, TEST2: second row, TEST3: third row)



Two dimensional tests (TEST1: first row, TEST2: second row, TEST3: third row)


T = 0
T = 0.005
T = 0.1
T = 0.05
T = 0.009
Solution
Space



TEST1: Two-dimensional surface plot of averaged solution at time $t = 0.001$ and its cross section at $y = 0.5$ for different times (first column). TEST2: (second column) and TEST3: third column

- Wick type SPDEs are easy to solve.
- The Wick approach is a robust tool for solving SPDEs
- SPDEs can be solved as actual PDEs and not only as integral equations.
- The Wick method is well defined for many multiplicative and non-linear SPDEs.
- Both SPDEs with additive or multiplicative noise can be solved numerically.

- Every thing in Numerical Methods for SPDEs has to be improved!
- Either the theory or the related computational methods for SPDEs are very poor in comparison to the deterministic one.
- We have to look for what is the **easiest and feasible** improvements.
- etc...

THANK YOU

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