جـامعـة الشـارقـة
UNIVERSITY OF SHARJAH

Symmetry-Driven Algorithms for Machine Learning

Application to the Human Anatomy

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Symmetry is an important feature of human anatomy and the absence of symmetry in medical images can indicate the presence of pathology.

Fast and effective quantification of symmetry in medical images for pathology detection: Application to chest radiography
Med Phys, 2017 Jun;44(6):2242-2256. doi: 10.1002/mp.12127. Epub 2017 May 4.

Laurens Hogeweg, Clara I Sanchez, Pragnya Maduskar, Rick H H M Philipsen, Bram van Ginneken

The physicians see the human body as reactions, connections, mutations, divisions ... of cells, chemicals and proteins, while mathematician see it as a combination of shapes and functions. Mostafa ZAHRI

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5 Implementation of ML for Symmetry detection
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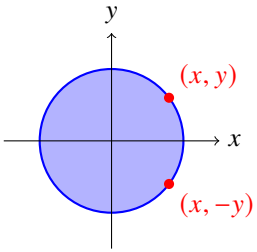
6 Accuracy measurement: Just Question

Definition

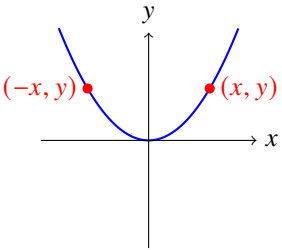
A graph or shape exhibits **symmetry with respect to the x-axis** if for every point (x, y), the point $(x, -y)$ is also on the shape. It exhibits **symmetry with respect to the y-axis** if for every point (x, y), the point $(-x, y)$ is also on the shape.

Example

The equation $x^2 + y^2 = 1$ (a circle) is symmetric about the x-axis.
The equation $y - x^2 = 0$ is symmetric about the y-axis.



Circle
Equation



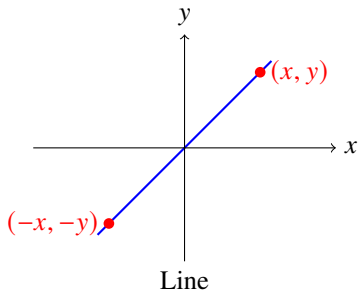
Parabola
Function

Definition

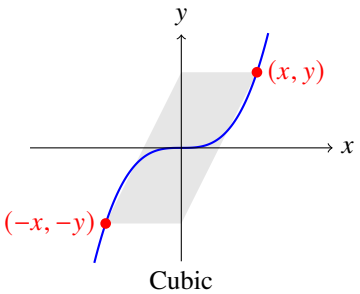
A graph or shape exhibits **symmetry with respect to the origin** if for every point (x, y), the point $(-x, -y)$ is also on the shape.

Example

The equation $y - x = 0$ is symmetric about the origin.
The equation $y - x^3 = 0$ is symmetric about the origin.



Line
Equation / Function



Cubic
Equation / Function

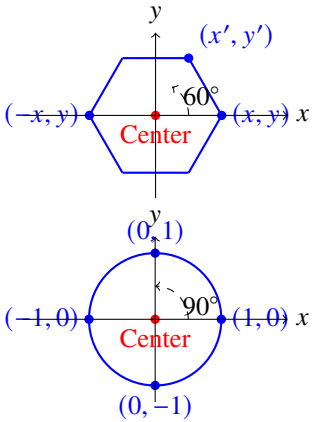
Definition

A shape exhibits **rotational symmetry** if it looks the same after being rotated about a center point by a certain angle less than 360°. The smallest angle for which this occurs is called the **order of rotation**. A circle has **infinite rotational symmetry**, as it looks identical after any rotation about its center.

Example

A regular hexagon has rotational symmetry of order 6, as it looks the same after being rotated by $60°, 120°, \ldots, 360°$.

The unit circle $(x^2 + y^2 = 1)$ remains unchanged under any rotation about the origin.



Definition

Rotational symmetry complements reflectional symmetry, offering additional geometric invariance.

Type	Test for Symmetry	Examples
x-axis	Replace y with $-y$; equation remains unchanged	$x^2 + y^2 = 1$
y-axis	Replace x with $-x$; equation remains unchanged	$y = x^2$
origin	Replace x with $-x$ and y with $-y$; unchanged	$y = x^3$
rotational	Rotate shape about a center point by θ; unchanged	Circle, Regular Hexagon

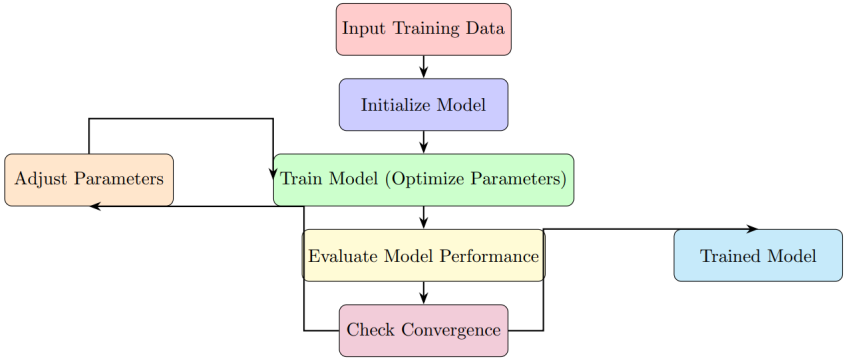
In polar coordinate; $b = (r, \theta)$ rotated by $\Delta\theta$

$$\Rightarrow \quad b' = (r, \theta + \Delta\theta)$$

In cartesian coordinate $b = (x, y)$ rotated by $\Delta\theta$

$$\Rightarrow \quad b' = (x', y') = (x\cos(\Delta\theta) - y\sin(\Delta\theta), x\sin(\Delta\theta) + y\cos(\Delta\theta))$$

Definition of Machine Learning Training refers to the process of teaching a machine learning model to perform a specific task by exposing it to a dataset. During training, the model learns patterns in the data by minimizing a loss function through optimization algorithms such as gradient descent. This process adjusts the model's parameters (weights and biases) to improve its predictions or classifications over time.



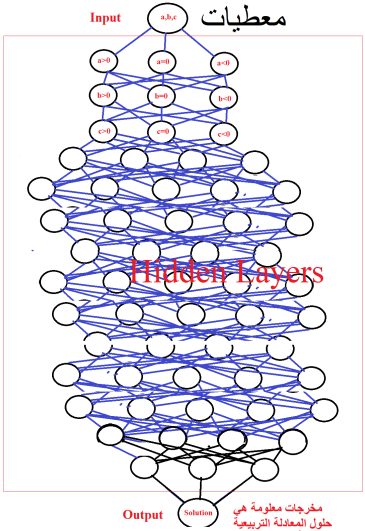
Training Data for Solving $ax^2 + bx + c = 0$ Without using the quadratic formula

Example	Input	Output
	(a, b, c)	
1	(1,-2, 1)	$x_0 = 1$ (One Solution)
2	(1,-5, 6)	$x_1 = 3, x_2 = 2$ (Two Solutions)
3	(1,1,1)	$\emptyset$ (No Real Solution)
4	(2,-4,2)	$x_0 = 1$ (One Solution)
⋮	⋮	⋮
100000	(3,1,1)	$x_1 = 0.43426, x_2 = -0.7675$ (Two Solutions)

Tabelle: Input and Output for Training Data

Given $(a, b, c) = (3, 4 - 5)$ estimate the solution of the related quadratic equation using AI.

Illustration of a Neural Network for solving the quadratic equation using AI



Algorithm:

1. Classify according sign first coef.
2. do same for next coef.
3. do same for next coef.
4. Classify according bigger or less 1
5. do same for next coef.
6. do same for next coef.

⋮

99. Solution estimated.

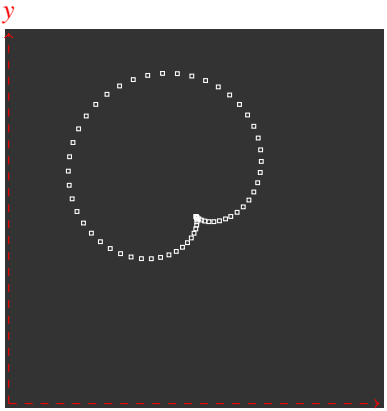
Problem: A solid curve that is not symmetric about the horizontal or vertical axes but gains symmetry after rotation.

Example: Consider the polar equation:

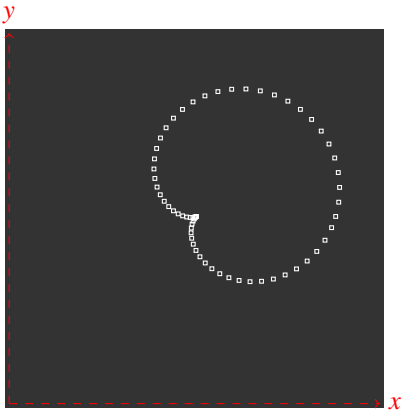
$$r(\theta) = 1 + \sin(\theta)$$

- Apple-like shape.
- No symmetry about the x- or y-axes.
- Gains symmetry after rotation by appropriate θ.

The parametric **Equation:** $r(\theta) = 1 + \sin(\theta)$ produces non-symmetric shapes depending on initial values. The pictures shows neither vertically nor horizontally symmetry. However, the shapes are symmetric !!!

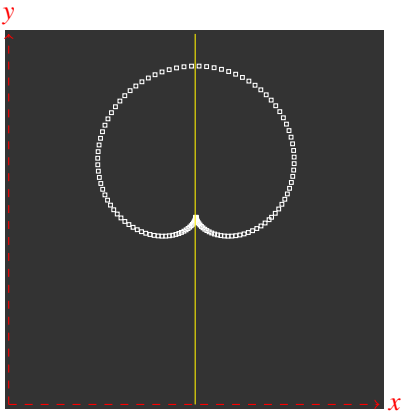


Capture for precise initial value θ_1 and low resolution.

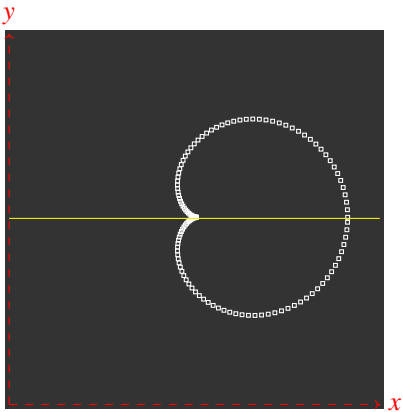


Capture for precise initial value θ_2 and low resolution.

One side symmetry: The parametric **Equation:** $r(\theta) = 1 + \sin(\theta)$ produces non-symmetric shapes depending on initial values. A suitable rotation (changing the initial values) leads to either a vertically or horizontally symmetry. This what we are looking for.

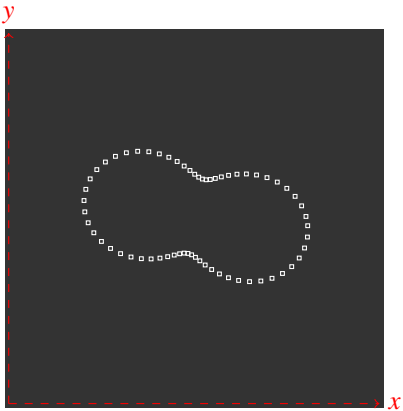


Vertical symmetry: High resolution due to linear interpolation

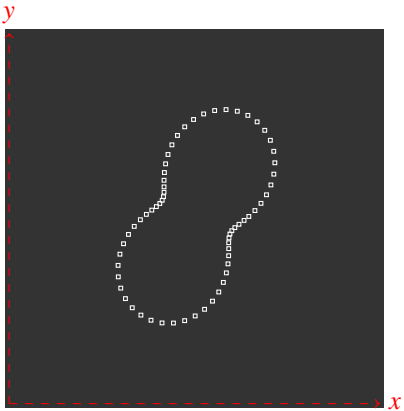


Horizontal symmetry: High resolution due to linear interpolation

The parametric **Equation:** $r(\theta) = 1 + \cos(\theta)\sin(\theta)$ produces non-symmetric shapes depending on initial values. The pictures shows neither vertically nor horizontally symmetry. However, the shapes are symmetric !!!

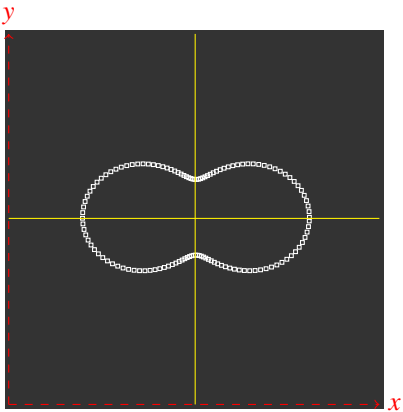


Capture for precise initial value θ_1:
Low resolution

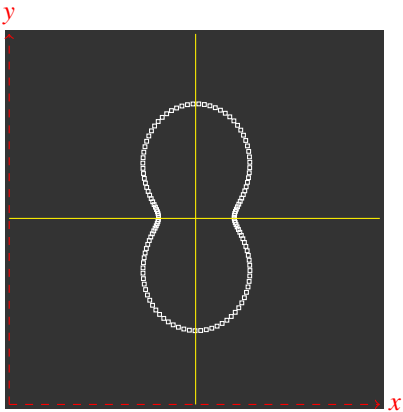


Capture for precise initial value θ_2:
Low resolution

Two sides symmetry: The parametric **Equation:** $r(\theta) = 1 + \sin(\theta)$ produces non-symmetric shapes depending on initial values. A suitable rotation (changing the initial values) leads to either a vertically or horizontally symmetry. This what we are looking for.

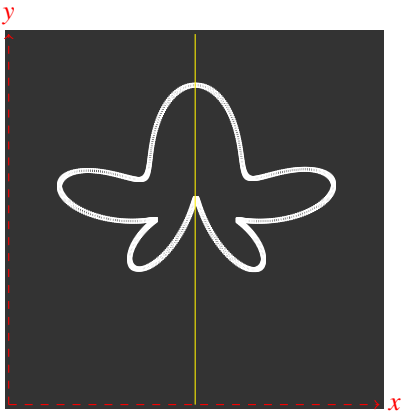


Vertical-Horizontal symmetry: Interpolation of nodes leads to High resolution

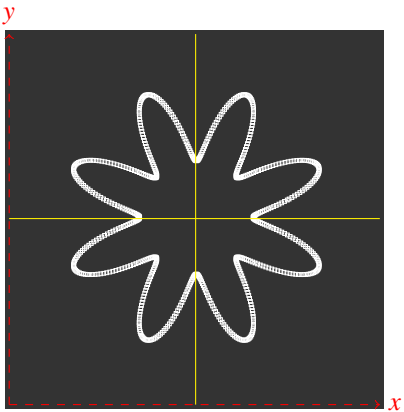


Vertical-Horizontal symmetry: Interpolation of nodes leads to High resolution

Multi-Symmetries or complex symmetries

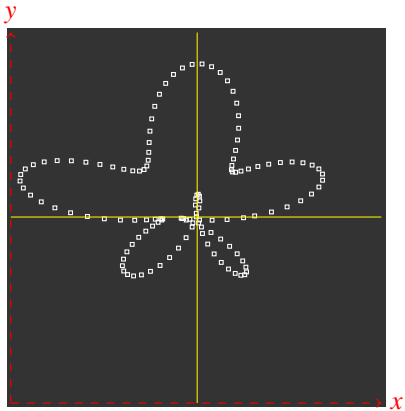


Vertical symmetry

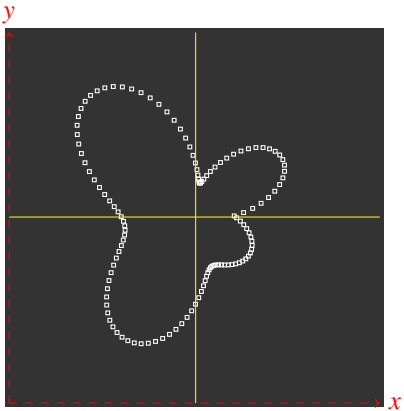


Horizontal-Vertical Symmetry

Absence of symmetries

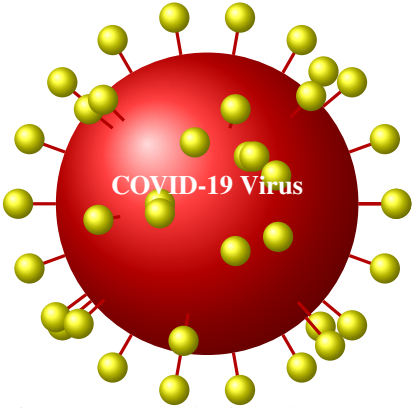


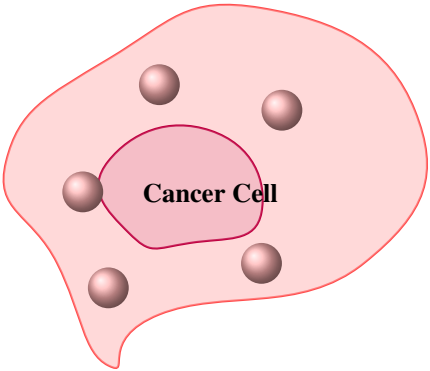
Absence of symmetry even after interpolation of nodes



Absence of symmetry even after interpolation of nodes

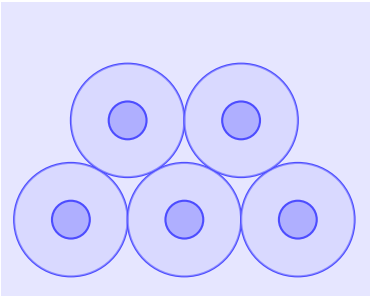
Absence of symmetries



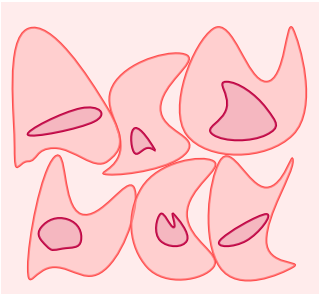


Pictures generated using ChatGPT.

Normal Cells vs. Cancer Cells: Absence of symmetries in Cancer cells



Normal Cells



Cancer Cells

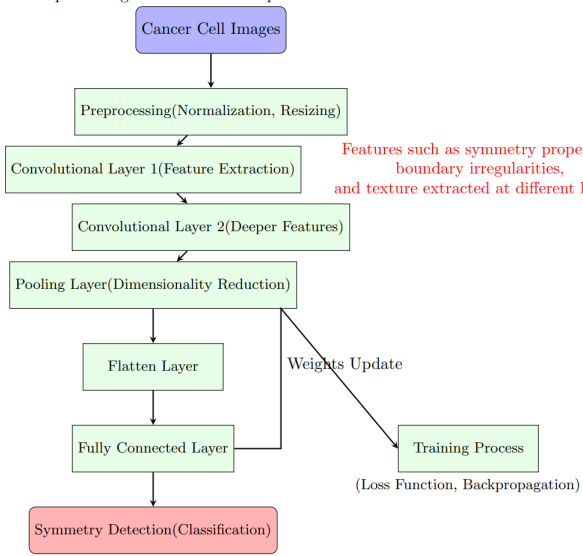
Normal Cells: Typically exhibit a high degree of symmetry, often spherical or elliptical, especially in tissues where structural integrity is crucial.

Cancer Cells: Often lose their regular symmetry and have irregular, asymmetrical shapes. This is due to:

See: Florent Peglio, and Sandrine Etienne-Manneville , *Cell polarity changes in cancer initiation and progression*, J. Cell Biol. 2024 Vol. 223 No. 1

Machine Learning Algorithm: Symmetry detection

Preprocessing ensures uniform input for the CNN.



Final output classifies cells as symmetric or asymmetric.

Let the sequence be $a_1, a_2, \ldots, a_n$. After t iterations, each element $A_k^{(t)}$ is computed as:

$$A_k^{(t)} = \begin{cases} \frac{1}{2}(A_{k+1}^{(t-1)} + A_k^{(t-1)}) & \text{if } k = 1, \text{(left boundary)} \\ \frac{1}{3}(A_{k-1}^{(t-1)} + A_k^{(t-1)} + A_{k+1}^{(t-1)}) & \text{if } 2 \leqslant k \leqslant n-1, \text{(middle elements)} \\ \frac{1}{2}(A_{k-1}^{(t-1)} + A_k^{(t-1)}) & \text{if } k = n, \text{(right boundary)} \end{cases}$$

For more precision, we can include more elements

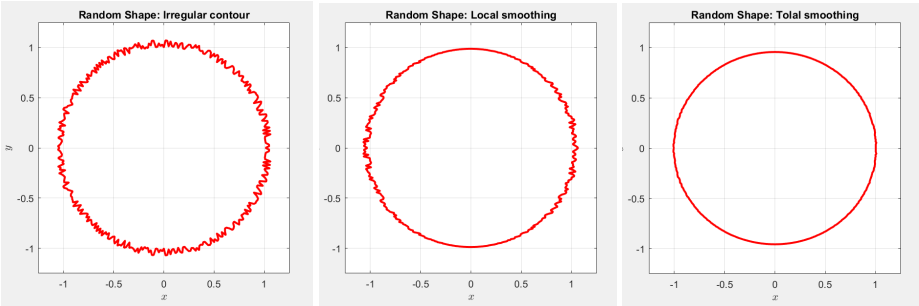
$$A_k^{(t)} = \begin{cases} \frac{1}{2}(A_{k+1}^{(t-1)} + A_k^{(t-1)}) & \text{if } k \leqslant 2, \text{(left boundary)} \\ \frac{1}{5}\left(A_{k-2}^{(t-2)} + A_{k-1}^{(t-1)} + A_k^{(t-1)} + A_{k+1}^{(t-1)} + A_{k+2}^{(t-1)}\right) & \text{if } 3 \leqslant k \leqslant n-2, \text{(middle elements)} \\ \frac{1}{2}(A_{k-1}^{(t-1)} + A_k^{(t-1)}) & \text{if } k \geqslant n-1, \text{(right boundary)} \end{cases}$$

Example: Let us consider the sequence $A = \{1, 4, -2, 5, 7, 2\}$.

1. Initial Sequence ($t = 0$):
$$A^{(0)} = \{1, 4, -2, 5, 7, 2\}.$$
2. After 5 Iterations ($t = 5$):
$$A^{(5)} = \{2.014, 2.208, 2.669, 3.265, 3.806, 4.063\}.$$
3. After 10 Iterations ($t = 10$):
$$A^{(10)} = \{2.436, 2.561, 2.832, 3.163, 3.440, 3.570\}.$$
4. After 100 Iterations ($t = 100$):
$$A^{(100)} = \{2.9987, 2.9990, 2.9996, 3.0004, 3.0010, 3.0013\}.$$

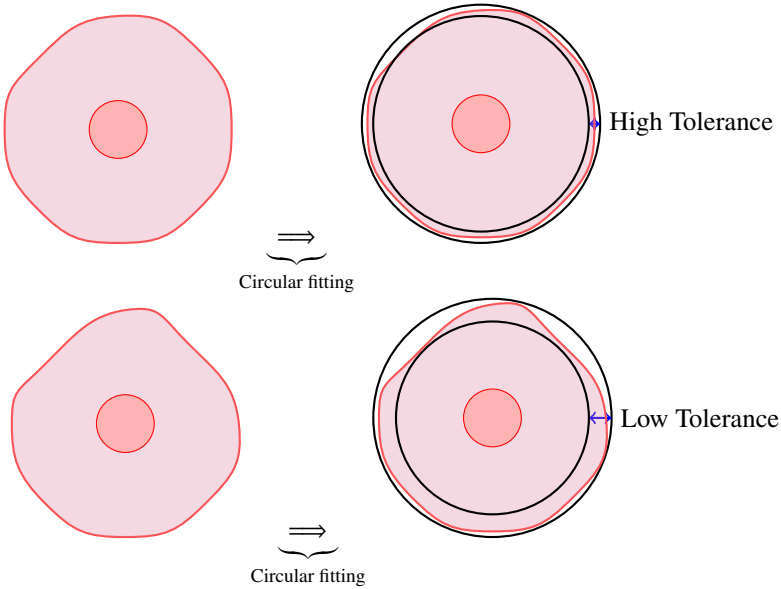
Remark: We can improve the averaging using more elements from left and from the right.

Repeated Average for a noisy curvature of a circle

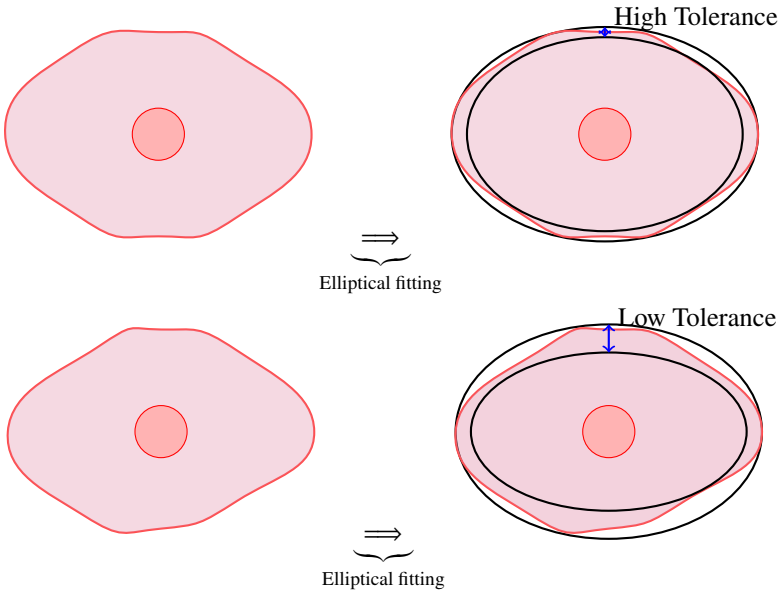


For $t = 0$ (left) for $t = 10$ middle picture and for $t = 100$ right picture.

Circular fitting of non-uniform cells between two circles



Elliptical fitting of non-uniform cells between two ellipses



Modeling of symmetrical shapes: Steady state of diffusing uniform distributed initial concentration on $(3, 3)$-decomposed (disjunct) domain:

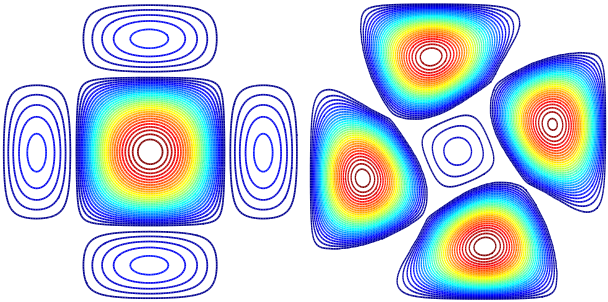


Fig.: Examples of symmetric diffusion.

Evolution of a deterministic concentration in time and space, $T > 0$,

Spatial domain $D \subset \mathbb{R}^2$ with Lipschitz boundaries ∂D,

u is the exact solution of following advection-diffusion problem (ADP):

$$du_t = [\nabla \cdot (d\nabla u)]dt - [a \cdot \nabla u]dt,$$

under Initial and Boundary conditions:

$$\begin{aligned} u(t, \mathbf{x}) &= 0, && (t, \mathbf{x}) \in [0, T] \times \partial D, \\ u(0, \mathbf{x}) &= \mathbf{f}(\mathbf{x}), && \mathbf{x} \in D, \end{aligned}$$

where $a \in (0, 1]^2$ is a domain dependent vector-valued velocity vector and d is the diffusion coefficient tensors with entries in $(0, 1]$.

under appropriate continuity and differentiability conditions on the initial solution u^0 in $L^2(D)$, there exists a unique solution u in $C([0, T], L^2(D)) \cap L^2(t_0, T; \mathbf{H}_0^1(D))$.

Construction of Interface Solution $\mathbf{w}^k$

For $(t, \mathbf{x}_{i,j}^{\beta}) \in [0,T] \times \Gamma_{k,h}$ Interface curves (indexed by β),

Continuity transmission: $\mathbf{w}^{k,h} \equiv \mathbf{w}^h \equiv \mathbf{w}^k$,

$\mathbf{w}^h$ boundary for D^h and $\mathbf{w}^k$ boundary for D^k,

Interpolation at t are results of one time-step backward:

$$\mathbf{w}^k(t, \mathbf{x}_{i,j}^{k,h}) := \sum_{\mathbf{x}_{i,j}^{\beta} \in S^k \cup S^h} \alpha_{i,j}^k u^k\left(t - \Delta t, \mathbf{x}_{i,j}^{\beta}\right),$$

where the set $S^{\beta}(i,j)$ represents the grid-points sets contain at least two grid point from a given finite difference discretization of the domain D.

How do we construct $\alpha_{i,j}^k$, for $k = 1, \ldots, 9$

Transforming the initial ADP into Advection-diffusion system,

Solving a decoupled semi linear system of the form:

$$d\mathbf{u}^k(t) = \mathbf{K}^k(t, \mathbf{u}^k(t))dt,$$

where $\mathbf{K}$ is a matrix with entries constructed from the time-space integration of the ADP. The new initial-boundary conditions are given by:

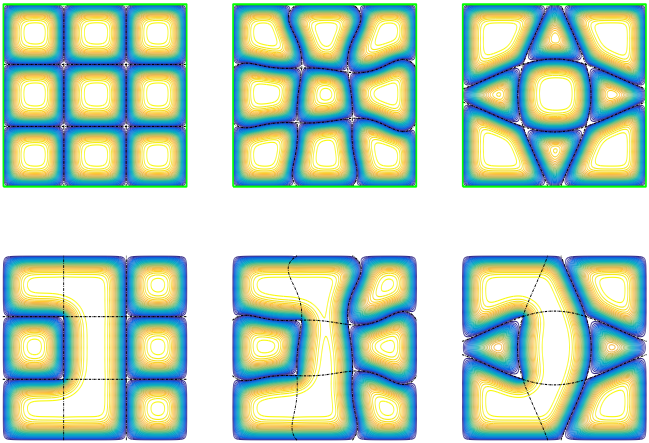
$$u_t^k\Big|_{\partial D_k \setminus \partial D} = \mathbf{w}^k\Big|_{\partial D_k \setminus \partial D},$$

where $k = 1, \ldots, 9$,

See M. Zahri, Modeling and Simulation of Symmetric Diffusion, *8th International Conference on Modeling Simulation and Applied Optimization (ICMSAO) IEEE Xplore: 24 October 2019 15-17 April 2019.* DOI: 10.1109/ICMSAO.2019.8880436

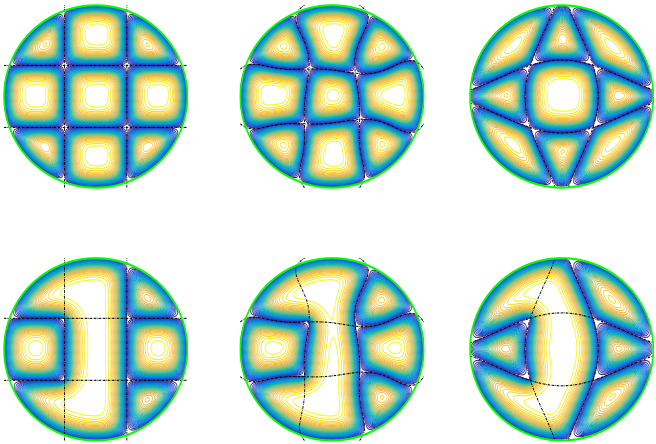
Symmetric shapes on a square domain

Diffusion of the local solutions on non-overlapping subdomains D_k for $k = 1, 2, \ldots, 9$,
Interface solutions considered as Dirichlet boundary conditions (i.e $\equiv 0$),
Simulation on decoupled subdomains:



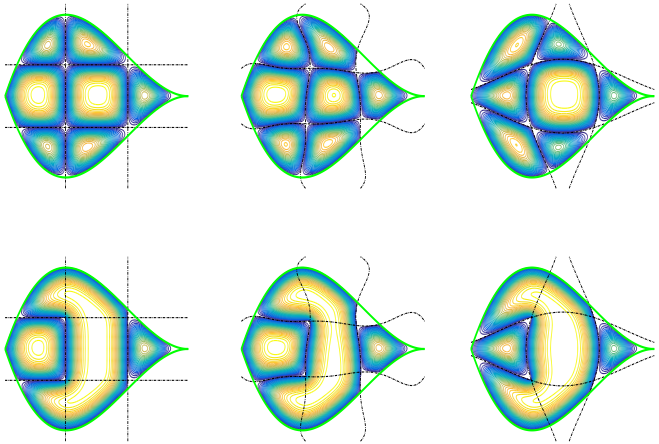
Symmetric shapes on a Disk domain

Diffusion of the local solutions on non-overlapping subdomains D_k for $k = 1, 2, \ldots, 9$,
Interface solutions considered as Dirichlet boundary conditions (i.e $\equiv 0$),
Simulation on decoupled subdomains:



Symmetric shapes on a Leaf-like domain

Diffusion of the local solutions on non-overlapping subdomains D_k for $k = 1, 2, \ldots, 9$,
Interface solutions considered as Dirichlet boundary conditions (i.e $\equiv 0$),
Simulation on decoupled subdomains:



Evolution of a non deterministic concentration in time and space, $T > 0$,

Spatial domain $D \subset \mathbb{R}^2$ with Lipschitz boundaries ∂D,

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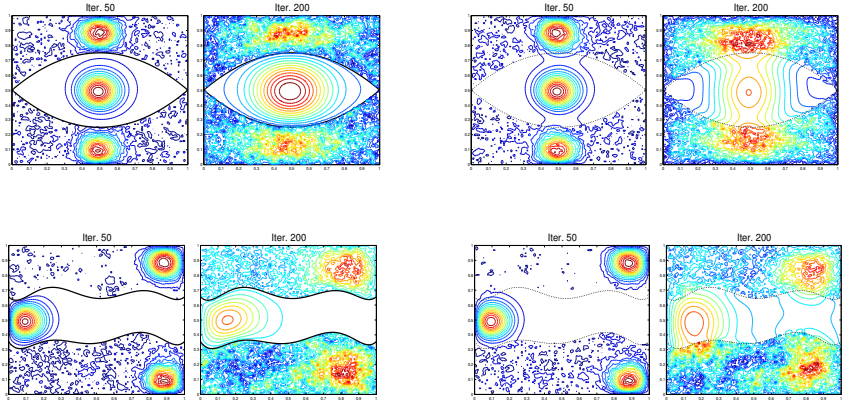
$$du_t = [\nabla \cdot (d\nabla u) - a \cdot \nabla u]dt + [g(u,t)]dW_t,$$

under (stochastic) Initial and Boundary conditions:

$$\begin{aligned} u(t,\mathbf{x},w) &= 0, && (t,\mathbf{x}) \in [0,T] \times \partial D, \\ u(0,\mathbf{x}) &= \mathbf{f}(\mathbf{x},w), && \mathbf{x} \in D, \end{aligned}$$

W_t refers to a Bownian motion

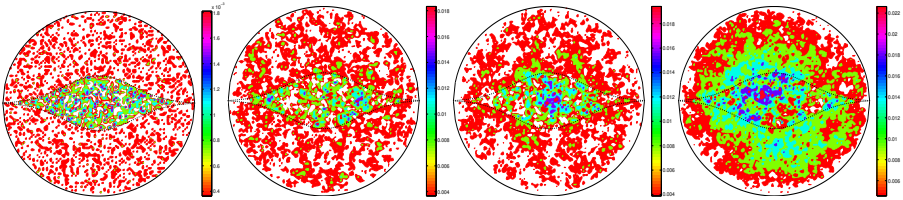
Convection-diffusion problem on Eye-like cut and Colon-like cut domains:



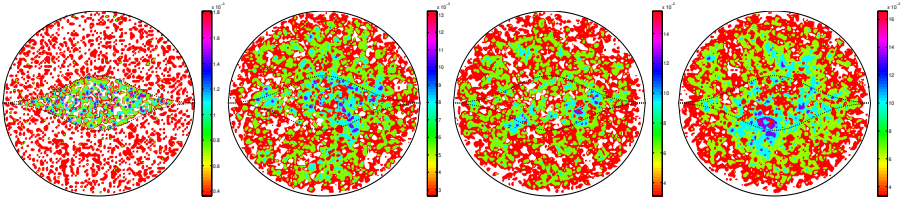
(left block) non-interpolated interface, (right block) interpolated interface,
This method (Separation of noises) leads to (stochastic) higher order of convergence.

M. Zahri, *Barycentric interpolation of interface solution for solving stochastic partial differential equations on non-overlapping subdomains with additive multi-noises* International Journal of Computer Mathematics, Volume 95(4),Pages 645-685, 2018

Spread of concentrations different types

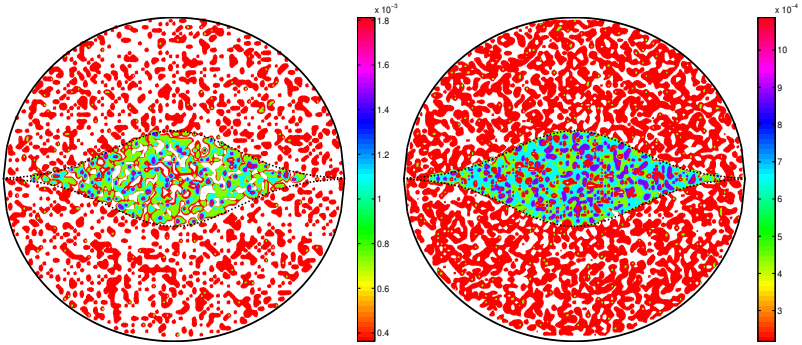


(A) A, $t = 0$ (B) A, $t = 100\Delta t$ (C) A, $t = 200\Delta t$ (D) A, $t = 5000\Delta t$



(E) B, $t = 0$ (F) B, $t = 100\Delta t$ (G) B, $t = 200\Delta t$ (H) B, $t = 5000\Delta t$

FIG.: SPDEs solution for tests A and B.



(A) $t = 0$ (B) $t = 5000\Delta t, M = 500$

- A. Rößler, M. Seaïd, M. Zahri, Numerical Simulation of Stochastic Replicator Models in Catalyzed RNA-like Polymers, MCS Vol. 79 (2009), Issue 12, P. 3577-3589.
- M. Zahri, Multidimensional Milstein scheme for solving a stochastic model for prebiotic evolution, JTUS, Vol. 8, Issue 2 (2014), P. 186-198.
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Definition

Supervised learning involves training a neural network using labeled datasets where each shape is paired with a symmetry label.

Steps:

1. **Dataset Preparation:** Collect labeled shapes with symmetry classes, e.g., x-axis, y-axis, rotational symmetry, etc.
2. **Model Design:** Use a Convolutional Neural Network (CNN) for images or feed-forward neural network for vectors.
3. **Training:** Train with a loss function (e.g., cross-entropy) and optimizer (e.g., Adam).
4. **Evaluation:** Test accuracy and metrics on a validation set.

Example Framework:

Input: Shape as an image.

Output: Symmetry class (e.g., x-axis, y-axis).

Definition

Unsupervised learning discovers symmetry patterns in shapes without labeled data.

Steps:

Dataset Preparation: Collect a set of unlabeled shapes.

Model Design: Use a generative model like an Autoencoder or Variational Auto-encoder (VAE).

Training: Optimize reconstruction loss to learn latent representations.

Evaluation: Visualize clusters or latent space to identify symmetry patterns.

Example Framework:

Input: Shape as an image.

Output: Latent representation or cluster corresponding to a symmetry type.

Definition

Reinforcement learning trains an agent to identify or generate symmetric shapes by interacting with an environment and receiving rewards.

Steps:

Environment Design: Define states as shapes and actions as symmetry transformations (e.g., reflections, rotations).

Reward Function:

Positive reward for correct symmetry identification or generation.

Negative reward for incorrect actions.

Model Design: Use Deep Q-Networks (DQN) or Proximal Policy Optimization (PPO).

Training: Maximize cumulative reward to classify or enforce symmetry.

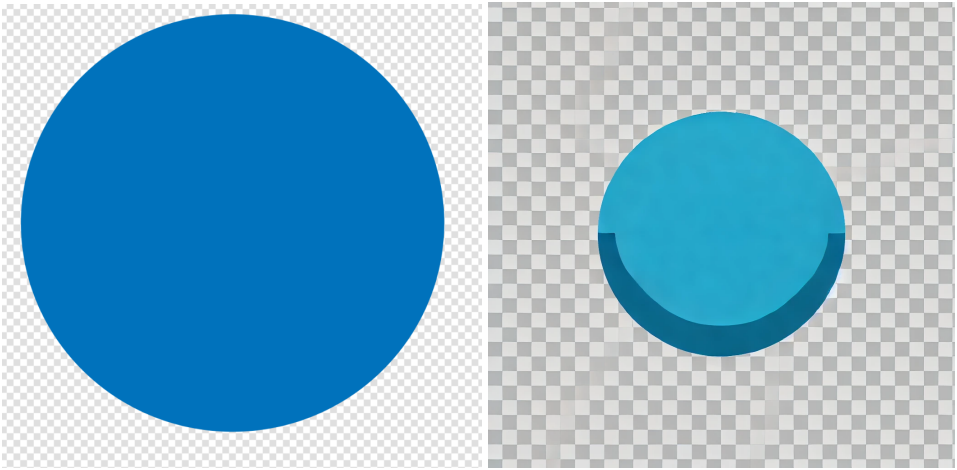
Evaluation: Test the agent on unseen shapes.

Example Framework:

Input: Shape and initial state.

Output: Sequence of actions to classify or enforce symmetry.

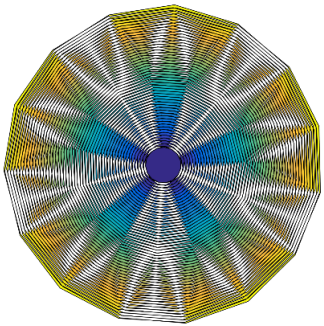
ChatGPT is asked to generate similar image. This is the answer



Input (left) – ChatPT-Output (right)

Possible explanation: Similarity and symmetry are acceptable.

ChatGPT is asked to generate similar image. This is the answer





Input (left) – ChatPT-Output (right)

Possible explanation: My Picture left is symmetrical due to the VERY large number of intersections of Chord lines. The result of these intersection gives a symmetrical **String Fractals**. However, the CNN uses finite number of layers to detect the similarities or the symmetry respectively.

References:

[1] Mostafa Zahri, **Barycentric interpolation of Interface Solution of a parallel domain decomposition method for solving Dirichlet IBV problems**, (Under review, see preprints in researchgate)

[2] Mostafa Zahri, **Barycentric interpolation of interface-solution for solving stochastic partial differential equations on non-overlapping subdomains with additive multi-noises**, International Journal of Computer Mathematics, Volume 95 (2018), Issue 4, Pages 645-685

[3] Mostafa Zahri, **Mathematical Modeling of a Class of Symmetrical Islamic Design**, Symmetry 2019, 11 (4), 517

THANK YOU

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