

Modeling and Simulation of Symmetric Diffusion

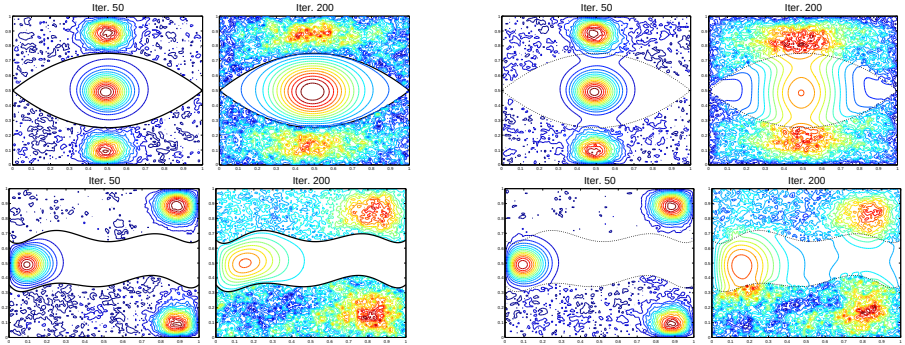
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Motivation

Domain Decomposition Solution of SPDEs: Barycentric interpolation Method (BIM) applied for solving Convection-diffusion problem on Eye-like cut and Colon-like cut domains:

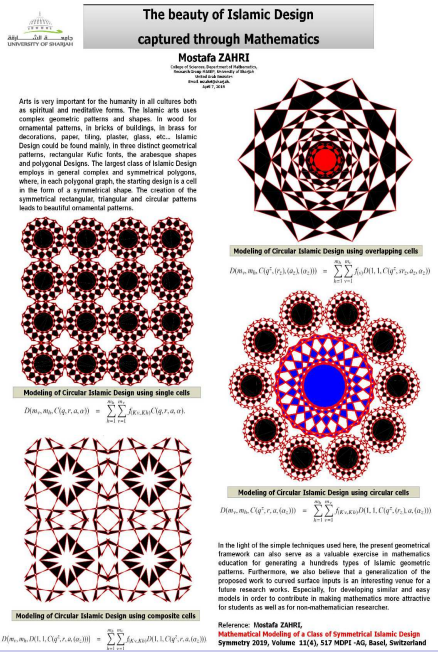


- (left block) non-interpolated interface, (right block) interpolated interface,
- This method (Separation of noises) leads to (stochastic) higher order of convergence.

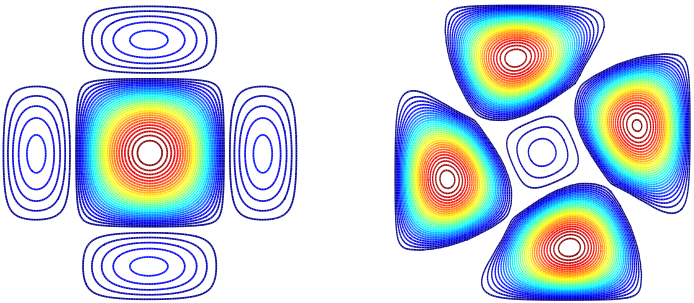
M. Zahri, *Barycentric interpolation of interface solution for solving stochastic partial differential equations on non-overlapping subdomains with additive multi-noises* International Journal of Computer Mathematics, Volume 95(4),Pages 645-685, 2018

Motivation, cont.

- The largest class of Islamic design employs in general symmetrical polygons,
- Each polygonal graph is the starting cell of a symmetrical shape,
- This Geometrical framework serve as a valuable exercise in mathematics education for generating hundreds types of Islamic geometric patterns,
- What about modeling of the symmetry aspect in nature !!!
- M. Zahri, *Mathematical Modeling of a Class of Symmetrical Islamic Design* Symmetry Volume 11 (4), pages 517, 2019



- Now combining BIM and Symmetry,
- Local and disjunct symmetrical diffusions,
- The limits behavior of diffusing uniform distributions on non-overlapped subdomains:



(left) Linear interfaces (Right) non-linear interfaces

- How about the construction of such shapes!!!
- In case of transitions, how is the Interface-Solution approximated!!!

Contents

- Formulation of symmetrical advection-diffusion
- Barycentric interpolation of interface solutions
- Numerical simulations symmetrical diffusion

Formulation of symmetric ADP

- For a positive time upper bound T,
- bounded domain $D \subset \mathbb{R}^2$ with Lipschitz boundaries ∂D,
- u is the exact solution of following advection-diffusion problem (ADP):

$$\frac{du}{dt} = \nabla \cdot (d\nabla u) - a \cdot \nabla u,$$

under Initial and Boundary conditions:

$$\begin{aligned} u(t, \mathbf{x}) &= 0, & (t, \mathbf{x}) \in [0, T] \times \partial D, \\ u(0, \mathbf{x}) &= \mathbf{f}(\mathbf{x}), & \mathbf{x} \in D, \end{aligned}$$

- where $a \in (0, 1]^2$ is a domain dependent vector-valued velocity vector and d is the diffusion coefficient tensors with entries in $(0, 1]$.
- under appropriate continuity and differentiability conditions on the initial solution u^0 in $L^2(D)$, there exists a unique solution u in $C([0, T], L^2(D)) \cap L^2(t_0, T; \mathbf{H}_0^1(D))$.

- Goal: Time-space integration of the ADP on non-overlapping decomposed domains $(D_k)_{1\leqslant k\leqslant 9}$ with smooth boundaries.
- Spatial step size $\Delta x = \Delta y$ for a square domain $D = [0,1]^2 \in \mathbb{R}^2$:

$$\begin{aligned}\frac{du_{i,j}^k(t)}{dt} &= d^k \frac{u_{i+1,j}^k(t) - 2u_{i,j}^k(t) + u_{i-1,j}^k(t)}{(\Delta x)^2} \\ &+ d^k \frac{u_{i,j+1}^k(t) - 2u_{i,j}^k(t) + u_{i,j-1}^k(t)}{(\Delta y)^2} \\ &- a^k \frac{u_{i+1,j}^k(t) - u_{i-1,j}^k(t)}{2\Delta x} \\ &- a^k \frac{u_{i,j+1}^k(t) - u_{i,j-1}(t)}{2\Delta y}.\end{aligned}$$

where d^k and a^k are the local advection-diffusion parameters.

- Transforming the initial ADP into Advection-diffusion system,
- Solving a decoupled semi linear system of the form:

$$d\mathbf{u}^k(t) = \mathbf{K}^k(t, \mathbf{u}^k(t))dt,$$

where $\mathbf{K}$ is a matrix with entries constructed from the time-space integration of the ADP. The new initial-boundary conditions are given by:

$$u_t^k\Big|_{\partial D_k \setminus \partial D} = \mathbf{w}^k\Big|_{\partial D_k \setminus \partial D},$$

where $k = 1, \ldots, 9$,

How do we approximate $\mathbf{w}^k$, for $k = 1, \ldots, 9$

- Construction of Interface Solution $\mathbf{w}^k$
- For $(t, \mathbf{x}_{i,j}^{\beta}) \in [0, T] \times \Gamma_{k,h}$ Interface curves (indexed by β),
- Continuity transmission: $\mathbf{w}^{k,h} \equiv \mathbf{w}^h \equiv \mathbf{w}^k$,
- $\mathbf{w}^h$ boundary for D^h and $\mathbf{w}^k$ boundary for D^k,
- Interpolation at t are results of one time-step backward:

$$\mathbf{w}^k(t, \mathbf{x}_{i,j}^{k,h}) := \sum_{\mathbf{x}_{i,j}^{\beta} \in S^k \cup S^h} \alpha_{i,j}^k u^k\left(t - \Delta t, \mathbf{x}_{i,j}^{\beta}\right),$$

where the set $S^{\beta}(i, j)$ represents the grid-points sets contain at least two grid point from a given finite difference discretization of the domain D.

How do we construct $\alpha_{i,j}^k$, for $k = 1, \ldots, 9$

Barycentric Interpolation Method, Convergence THM.

The weights $\alpha_{i,j}^{\beta}$ are nonnegative with values in $[0,1)$: $\sum_{\mathbf{x}_{i,j}^{\beta} \in S^h \cup S^h} \alpha_{i,j}^{\beta} = 1$.

Theorem (See Ref. 1)

Consider an interface point $\mathbf{x}_{i,j} \in \Gamma_{1,2}$. If the following holds $4d\Delta t < (\Delta x)^2$, $\Delta x < 2d/a$, then there exist a sequence $(\alpha_k)_k$, such that following two step Barycentric scheme

$$U_{i,j}^{n+1} = \Phi(U_{i-2,j}^{1,n}, U_{i-1,j}^{1,n}, U_{i,j-2}^{1,n}, U_{i,j-1}^{1,n}, U_{i+1,j}^{2,n}, U_{i+2,j}^{2,n}, U_{i,j+1}^{2,n}, U_{i,j+2}^{2,n}). \tag{1}$$

approximate $\boldsymbol{U}(\mathbf{x}_{i,j}, t_{n+1})$ in the two-dimensional ADP and in two-dimensional AD-System. Moreover, the convergence orders in time and space are preserved. Namely, the first order in time $O(\Delta t)$ and the second order $O((\Delta x)^2)$ in space. Where Φ represents the Barycentric mapping.

The proof idea:

$$\begin{aligned}\Phi(\cdots) = U_{i,j}^{n+1} \quad = \quad & \frac{(1-2\alpha)}{4}(U_{i-2,j}^{n} + U_{i+2,j}^{n} + U_{i,j-2}^{n} + U_{i,j+2}^{n}) \\ & + \frac{\alpha}{2}\boldsymbol{U}_{i-1,j}^{n} + \frac{\alpha}{2}\beta^{+}U_{i+1,j}^{n} + \frac{\alpha}{2}\boldsymbol{U}_{i,j-1}^{n} + \frac{\alpha}{2}\beta^{+}U_{i,j+1}^{n}\end{aligned}$$

where $\alpha = \dfrac{2d\Delta t}{(\Delta x)^2}$, $\quad \beta^{-} = 1 - \dfrac{a}{2d}\Delta x$, $\quad$ and $\quad \beta^{+} = 1 + \dfrac{a}{2d}\Delta x$.

DDM-BIM Algorithm

DDM-BIM Algorithm:

Step 0 *Fix a symmetrical DD:* $D = \bigcup_{k=1}^{r} D_k$
Fix $\Delta x = \Delta y$.
Fix the interface of the symmetrical DD.
Fix the IC u_0 *on the whole domain D.*
Set the IC (time=0) for each subdomain
D_k *for* $k = 1, \ldots, r$.
Set the BC for the whole domain D

Step 1 *Set up similar interpolation function*
of the dynamical internal boundary conditions.

Step 2 *Compute the interface solution*
$\boldsymbol{w}^k(1, \cdot)$ *using* u_0 *and BIM-Algorithm.*

Step 3 *Compute one time-iteration for* u_1^k
for $k = 1, \ldots, r$ *by using the time loop*

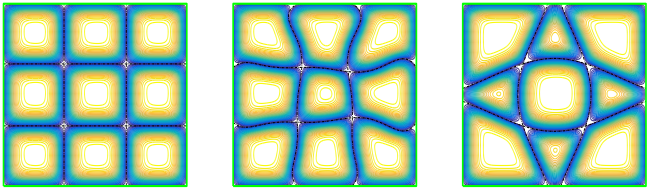
Step 4 *Update the solution*
(collecting interior and interface solutions).

Step 5 *in step 0, replace* u_0 *by* u_1
and repeat the steps 1. to 5. to compute u_2

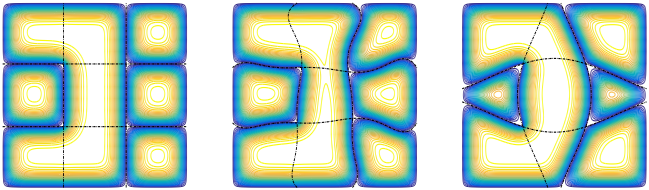
Step 6 *Repeat 1-5 until desired stop-time.*

Symmetric shapes on a square domain

- Diffusion of the local solutions on non-overlapping subdomains D_k for $k = 1, 2, \ldots, 9$,
- Interface solutions considered as Dirichlet boundary conditions (i.e $\equiv 0$),
- Simulation on decoupled subdomains:

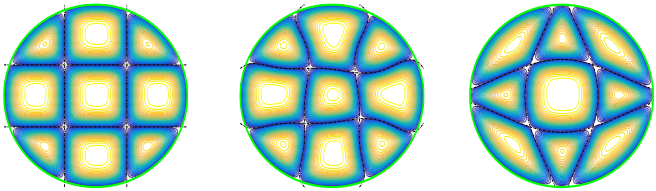


- Diffusion of the local solutions on non-overlapping subdomains D_k for $k = 1, 2, \ldots, 9$.
- Interpolate the interfaces $\Gamma_{1,2}$, $\Gamma_{2,5}$, $\Gamma_{5,8}$, and $\Gamma_{7,8}$, using the *BIM* method:

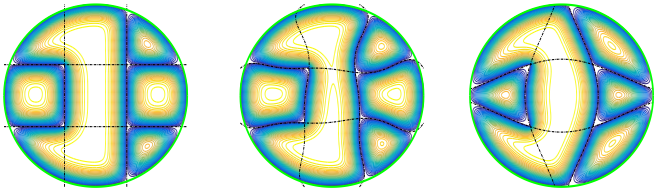


Symmetric shapes on a Disk domain

- Diffusion of the local solutions on non-overlapping subdomains D_k for $k = 1, 2, \ldots, 9$,
- Interface solutions considered as Dirichlet boundary conditions (i.e $\equiv 0$),
- Simulation on decoupled subdomains:

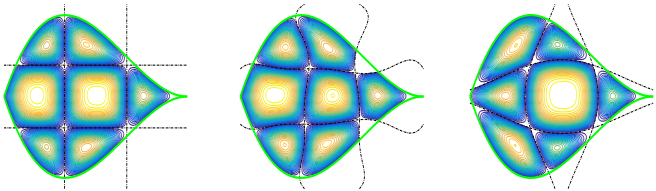


- Diffusion of the local solutions on non-overlapping subdomains D_k for $k = 1, 2, \ldots, 9$,
- Interpolate the interfaces $\Gamma_{1,2}$, $\Gamma_{2,5}$, $\Gamma_{5,8}$, and $\Gamma_{7,8}$, using the *BIM* method:

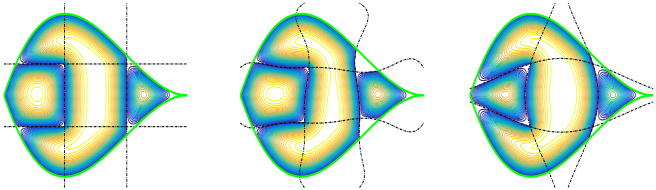


Symmetric shapes on a Leaf-like domain

- Diffusion of the local solutions on non-overlapping subdomains D_k for $k = 1, 2, \ldots, 9$,
- Interface solutions considered as Dirichlet boundary conditions (i.e $\equiv 0$),
- Simulation on decoupled subdomains:



- Diffusion of the local solutions on non-overlapping subdomains D_k for $k = 1, 2, \ldots, 9$,
- Interpolate the interfaces $\Gamma_{1,2}$, $\Gamma_{2,5}$, $\Gamma_{5,8}$, and $\Gamma_{7,8}$, using the *BIM* method:



- Simulation of Symmetrical Phenomena.
- Practical use for generating geometric patterns and shapes: Mathematics and Arts.
- Numerical Simulation on non regular domains.
- Construction of higher order schemes for SPDEs (Separating noises).
- . . .

References:

[1] Mostafa Zahri, **Barycentric interpolation of Interface Solution of a parallel domain decomposition method for solving Dirichlet IBV problems**, (Under review, see preprints in researchgate)

[2] Mostafa Zahri, **Barycentric interpolation of interface-solution for solving stochastic partial differential equations on non-overlapping subdomains with additive multi-noises**, International Journal of Computer Mathematics, Volume 95 (2018), Issue 4, Pages 645-685

[3] Mostafa Zahri, **Mathematical Modeling of a Class of Symmetrical Islamic Design**, Symmetry 2019, 11 (4), 517

THANK YOU

MOSTAFA ZAHRI