

Numerical Methods for Stochastic Differential Equations

Computational Approach

Mostafa Zahri

Department of Mathematics, Research Group MASEP
University of Sharjah
United Arab Emirates

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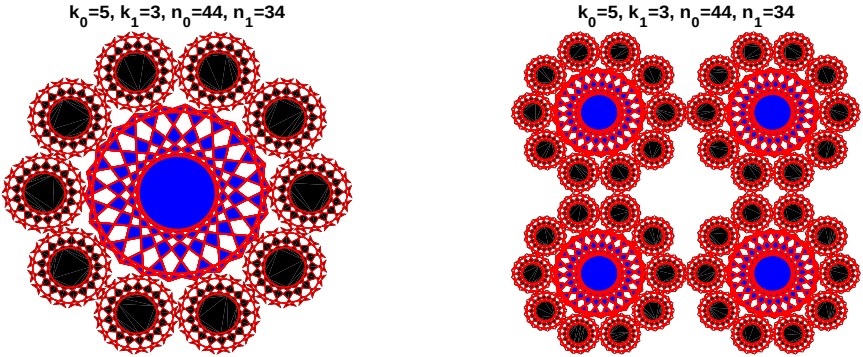
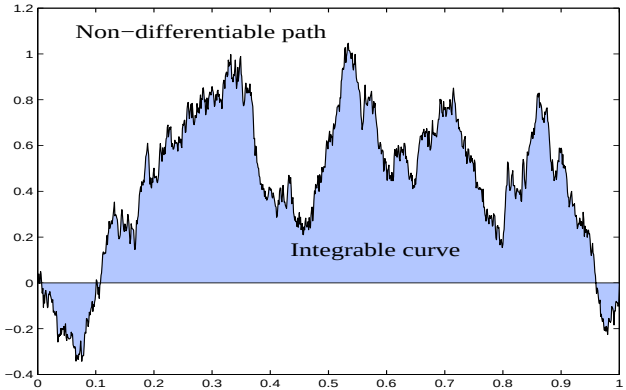


ABBILDUNG: A beautiful side of mathematics. Mathematics and Arts

Numerical Schemes for SDEs

- Itô Taylor-Expansion
- Itô Taylor Family Schemes (Strong and Weak convergence)
- Analysis of Itô Taylor Schemes, Computational Examples

The Non smooth path bellow i continuous **BUT nowhere differentiable**. The the use of the deterministic methods is not possible. (Mathematical problem): **Integration instead of differentiation**



The figure above shows a nowhere differentiable stochastic process and a bounded area under the curve.

Therefore integration instead of differentiation

Deterministic vs stochastic Expansion

- The Taylor expansion of a real function $f(x)$ that is infinitely differentiable at a real x_0 is

$$f(x) = \underbrace{\sum_{j\in\mathcal{A}} \frac{(x-x_0)^j}{j!} f^{(j)}(x_0)}_{Approximation\ Term\ P_n(x)} + \underbrace{\sum_{j\in\mathcal{B}(\mathcal{A})} \frac{(x-x_0)^j}{j!} f^{(j)}(x_0)}_{Remainder\ Term} \tag{1}$$

  where $\mathcal{A} := \{0, 1, \ldots, p\}$ and $\mathcal{B}(\mathcal{A})$ represents the 'classical' complement of $\mathcal{A}$ in $\mathbb{N}_0$.

- The stochastic Taylor-Expansion of a real valued stochastic process function $f(t, X_t)$ at measurable point (τ, X_τ)

$$f(\tau, X_\tau) = \underbrace{\sum_{\alpha\in\mathcal{A}} I_\alpha[f_\alpha(\rho, X_\rho)]}_{Approximation\ Term} + \underbrace{\sum_{\alpha\in\mathcal{B}(\mathcal{A})} I_\alpha[f_\alpha(., X.)]_{\rho,\tau},}_{Remainder\ Term} \tag{2}$$

  where ρ and τ are Stop-times satisfying ..., X_t is an adapted Itô process, $\mathcal{A}$ is an index set of the main approximation term and $\mathcal{B}(\mathcal{A})$ represents the complement of $\mathcal{A}$.

Consider the Itô SDE in $\mathbb{R}^d$ for $t \in [0, T]$

$$dX_t = a(t, X_t)dt + \sum_{j=1}^{m} b^j(t, X_t)dW_t^j, \tag{3}$$

with

- drift and diffusion coefficients $a, b^j : \mathbb{R}^d \to \mathbb{R}^d$ for $j = 1, \ldots, m$
- an m-dimensional Wiener process $W_t = (W_t^1, \ldots, W_t^m$

This is really an Itô stochastic **integral equation**

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \sum_{j=1}^{m} \int_0^t b^j(s, X_s)dW_s^j, \tag{4}$$

For $Y_t = f(t, X_t)$ consider the one dimensional SDE

$$dY_t = a(t, X_t)dt + b(t, X_t)dW_t$$

The one dimensional Itô Formula is given by

$$\boxed{dY_t = L^0 f(t, X_t)dt + L^1 f(t, X_t)dW_t,} \tag{5}$$

Where the differential operators L^0 and L^1 are given as follows:

$$L^0 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \underbrace{\frac{1}{2}b^2\frac{\partial^2}{\partial x^2}}_{\text{due to } E(\Delta W_t)=\Delta t} \qquad \text{and} \qquad L^1 = b\frac{\partial}{\partial x}.$$

Consider a one dimensional Brownian motion and a non-necessary uniform time discretization $t_k = k\frac{T-t_0}{2^n}$ of the interval $[t_0, T]$. Then we have,

$$1). \quad \lim_{n\to\infty} \sum_{k=0}^{2^n-1} (\Delta t_k)^2 = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \Delta t_k \Delta W_{t_k} = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \Delta W_{t_k} \Delta t_k = 0.$$

$$2). \quad \lim_{n\to\infty} \sum_{k=0}^{2^n-1} (\Delta W_{t_k})^2 = \int_{t_0}^{T} ds = (T - t_0). \qquad \text{(Convergence in } L^2\text{)}$$

where $\Delta t_k = t_{k+1} - t_k$ and $\Delta W_{t_k} = W_{t_{k+1}} - W_{t_k}$.
Per passage to the continuous case, this means

$$\boxed{dtdt = dtdW_t = dW_t dt = 0},$$

$$\boxed{dW_t dW_t = dt.}$$

Using the definition of the Itô, the following are one dimensional Itô integral (OSI):

$$\int_0^t dW_s = \lim_n \sum_{i=0}^{n} \left(W_{t_{i+1}} - W_{t_i}\right) = W_t - W_0$$

$$\int_0^t W_s dW_s = \lim_n \sum_{i=0}^{n} W_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)$$

$$= \lim_n \sum_{i=0}^{n} W_{t_i} \Delta W_{t_i}$$

$$\boxed{(W_t)^2 - 2W_{t_i}\Delta W_{t_i} = \sum_{i=0}^{n} \left(\Delta W_{t_i}\right)^2}$$

$$= \frac{1}{2}(W_t)^2 - \frac{1}{2} \underbrace{\lim_n \sum_{i=0}^{n} \left(\Delta W_{t_i}\right)^2}_{\to t \quad (\text{in } L^2)} \qquad \text{later !!!}$$

$$= \frac{1}{2}(W_t)^2 - \frac{1}{2}t$$

Clearly the behavior of this integral is different from the Riemann-Stieltjes

One dimensional Itô formula: Example

For $f(t,x) = \frac{1}{2}x^2$ with $X_t = W_t$ and $a_t = 0, b_t = 1$. Applying Itô formula:

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + a_t\frac{\partial f}{\partial x}dt + b_t\frac{\partial f}{\partial x}dW + \frac{1}{2}b_t^2\frac{\partial^2 f}{\partial x^2}dt \\ &= \frac{\partial f}{\partial t}dt + (1)\frac{\partial f}{\partial x}dW + \frac{1}{2}(1)^2\frac{\partial^2 f}{\partial x^2}dt \\ &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dW + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt \\ &= \frac{\partial f}{\partial x}dW + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt \end{aligned}$$

Hence $\frac{1}{2}dW_t^2 = W_t dW_t + \frac{1}{2}dt$ and $\frac{1}{2}\int dW_s^2 = \int W_s dW_s + \frac{1}{2}\int dt$ Thus,

$$\boxed{I_t = \int_0^t W_s dW_s = \frac{1}{2}(W_t^2 - t)}$$

We got the same result as by using the variational calculus.

For $m \in \mathbb{N}$ and $j = 1, 2, \ldots, m$, we associate a B.M. W^j and for $j = 0$, we associate the deterministic variable t . For $\alpha = (j_1, \ldots, j_l)$. We denote the **size** and the **number of zero components**of α

$$\begin{aligned} \ell(\alpha) &:= \ell_\alpha \qquad number\ of\ components\ of\ \alpha \\ n(\alpha) &:= n_\alpha \qquad number\ of\ zero\ components\ of\ \alpha \end{aligned}$$

The set of all multi-indices with respect to $F = \{0, 1, \ldots, m\}$ is represented by

$$\mathcal{M} = \bigcup_{l=1}^{\infty} F^l \cup \{v\}, \tag{6}$$

where v refers to the empty multi-index with size zero. The following example gives more sense for the definition above:

EXAMPLE

For $\alpha = (1, 0, 2)$ it holds $\ell(\alpha) = 3 \quad$ and $\quad n(\alpha) = 1$.
For $\alpha = (1, 0, 0, 2, 3, 1, 0, 0)$ it holds $(\alpha) = 8 \quad$ and $\quad n(\alpha) = 4$.

Let us consider $\alpha = (j_1, j_2, \ldots, j_l)$ a multi-index and $(W_t)_{t\geqslant 0}$ an m-dim BM. For a square integrable function f, the Multiple Itô-Integrals are defined per recursion as follows:

$$I_\alpha[f(.)]_{t_0,t} := \begin{cases} f(t), & \text{if } l = 0 \quad (i.e. \alpha = \nu) \\ \int_{t_0}^{t} I_{\alpha-}[f(.)]_{t_0,s} ds, & \text{if } l \geqslant 1 \text{ and } j_l = 0 \\ \int_{t_0}^{t} I_{\alpha-}[f(.)]_{t_0,s} dW_s^{j_l}, & \text{if } l \geqslant 1 \text{ and } j_l \geqslant 1. \end{cases}$$

Where $\alpha- := (j_1, j_2, \ldots, j_{l-1})$ drop the last component. **Examples**:

$$\begin{aligned} W_t^0 &= t \quad I_{\alpha,t} = I_\alpha[1]_{0,t} \\ I_{(1,2)}[f(.)]_{t_0,t} &= \int_{t_0}^{t}\int_{t_0}^{s} f(z) dW_z^1 dW_s^2, \\ I_{(1,2,0)}[f(.)]_{t_0,t} &= \int_{t_0}^{t} I_{(1,2)}[f(.)]_{0,s} ds = \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{s_1} f(s_2) dW_{s_2}^1 dW_{s_1}^2 ds. \end{aligned}$$

Consider the one dimensional Itô-Process X_t, which satisfy the Itô-Integral equation:

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)ds + \int_{t_0}^{t} b(X_s)dW_s \tag{7}$$

where a and b reel valued smooth functions, satisfying time-growth condition and $f \in C^2(\mathbb{R}, \mathbb{R})$. By applying the Itô Formula for $f(X_s)$, it follows:

$$\begin{aligned} f(X_s) &= f(X_{t_0}) + \int_{t_0}^{t} \left(a(X_s)\frac{\partial}{\partial x} f(X_s) + \frac{1}{2} b^2(X_s)\frac{\partial^2}{\partial x^2} f(X_s) \right) ds \\ &\quad + \int_{t_0}^{t} b(X_s)\frac{\partial}{\partial x} f(X_s) dW_s \\ &= f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s) ds + \int_{t_0}^{t} L^1 f(X_s) dW_s. \end{aligned}$$

If $f(x) = x \in \mathbb{R}$ then $L^0 f = a$ and $L^1 f = b$ the Itô-Taylor Expansion is given in integral form as

$$\begin{aligned} X_t &= X_{t_0} + \int_{t_0}^{t} \left(a(X_{t_0}) + \int_{t_0}^{s} L^0 a(X_z) dz + \int_{t_0}^{s} L^1 a(X_z) dW_z \right) ds \\ &\quad + \int_{t_0}^{t} \left(b(X_{t_0}) + \int_{t_0}^{s} L^0 b(X_z) dz + \int_{t_0}^{s} L^1 b(X_z) dW_z \right) dW_s \\ &= X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s + \underbrace{R_1}_{Remainder} \end{aligned}$$

where the remainder term is given by:

$$\begin{aligned} R_1 &= \int_{t_0}^{t} \int_{t_0}^{s} L^0 a(X_z) dz ds + \int_{t_0}^{t} \int_{t_0}^{s} L^1 a(X_z) dW_z ds \\ &\quad + \int_{t_0}^{t} \int_{t_0}^{s} L^0 b(X_z) dz dW_s + \int_{t_0}^{t} \int_{t_0}^{s} \underbrace{L^1 b(X_z)}_{Expand\ next} \, dW_z dW_s. \end{aligned}$$

By applying the Itô-Formula to the function $L^1 b$, we get(Milstein schemes)

$$\begin{aligned} X_t &= X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s \\ &\quad + L^1 b(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dW_z dW_s + \underbrace{R_2}_{Remainder} \end{aligned}$$

where the remainder term is given by:

$$\begin{aligned} R_2 &= \int_{t_0}^{t} \int_{t_0}^{s} L^0 a(X_z) dz ds + \int_{t_0}^{t} \int_{t_0}^{s} L^1 a(X_z) dW_z ds \\ &\quad + \int_{t_0}^{t} \int_{t_0}^{s} L^0 b(X_z) dz dW_s + \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^0 L^1 b(X_u) du dW_z dW_s \\ &\quad + \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^1 \underbrace{L^1 b(X_u)}_{Expand\ next} dW_u dW_z dW_s. \end{aligned}$$

Where $dtdt = dtdW_t = dW_t dt = 0$ and $dW_t dW_t = dt$ (in L^2 i.e in the mean sense)

A set $\mathcal{A}$ of Multi-Index is called **Hierarchical Index-Set**, if the following condition are satisfied

- $\mathcal{A} \neq \emptyset$,
- $\ell_m = \sup_{\alpha \in \mathcal{A}} \quad l(\alpha) < \infty$,
- $-\alpha \in \mathcal{A}; \quad \forall\, \alpha \in \mathcal{A} \backslash \{v\}$.

For any hierarchical set $\mathcal{A}$, we define the corresponding **Remainder Index-Set** as:

$$\mathcal{B}(\mathcal{A}) = \{\alpha \in \mathcal{M} \backslash \{\mathcal{A}\} : -\alpha \in \mathcal{A}\}.$$

Theorem

Let ρ and τ are two Stop-times such that $t_0 \leqslant \rho(w) \leqslant \tau(w) \leqslant T$. If the function $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}$ and all its differential are smooth and if the partial differential of a and b exist, then for any set $\mathcal{A} \in \mathcal{M}$ follow the stochastic Taylor-Expansion

$$f(\tau, X_\tau) = \sum_{\alpha \in \mathcal{A}_\gamma} I_\alpha[f_\alpha(\rho, X_\rho)] + \sum_{\alpha \in \mathcal{B}(\mathcal{A}_\gamma)} I_\alpha[f_\alpha(., X.)]_{\rho,\tau}, \tag{8}$$

Where the **Itô coefficient functions** are defined by

$$f_\alpha = \begin{cases} f & \text{if} \quad \ell(\alpha) = 0; \\ L^{j_1} f_{-\alpha} & \text{if} \quad \ell(\alpha) > 0. \end{cases}$$

for $\alpha = (j_1, \ldots, j_l) \in M$ and $f \in C^{n(\alpha), \ell(\alpha)+n(\alpha)}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R})$.

Theorem

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Where the **Itô coefficient functions** are defined by

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for $\alpha = (j_1, \ldots, j_l) \in M$ and $f \in C^{n(\alpha), \ell(\alpha)+n(\alpha)}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R})$.

In the EM scheme, the numerical solution to the SDE, Y_t converges to the exact solution X_t. We have to distinct between the following types of convergence :

- (**Path Approximation**). We said that the c**onvergence is strong** of order γ if
$$\lim_{N_T \to \infty} E\, s(\gamma) := \lim_{N_T \to \infty} E\left(|Y_{N_T}^{\gamma} - X_t|\right) = 0.$$
- (**Distributional Approximation**). We said that the **convergence is weak** of order γ if
$$\lim_{N_T \to \infty} E\, w(\gamma) := \lim_{N_T \to \infty} \left| E\left(g\left(Y_{N_T}^{\gamma}\right)\right) - E\left(g(X_t)\right)\right| = 0.$$

Where N_T refers the number of the time steps used for the approximation, g any polynomial function.

For $l \in \mathbb{N}$ and $\alpha = (j_1, \ldots, j_l)$ it exists a positive constant K independent from Δ such that

$$\boxed{\left(\boldsymbol{E}(I^2_{\alpha,\Delta})\right)^{\frac{1}{2}} \leqslant K\Delta^{\frac{\ell(\alpha)+n(\alpha)}{2}}} \tag{10}$$

Moreover, for X a d-dim stoch. process sol. of SDE, a multi-Index $\alpha \in \mathcal{M}$ and $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}$ satisfying If $|f_\alpha(t,x)|^2 \leqslant K\left(1 + |x|^2\right)$, the we have

$$\boxed{\boldsymbol{E}\left|I_\alpha\left[f_\alpha(t, X_t)\right]_{0,\Delta}\right|^2 \leqslant K\left(1 + \boldsymbol{E}|X_t|^2\right)\Delta^{l(\alpha)+n(\alpha)}} \tag{11}$$

Remark that the real number $\frac{\ell(\alpha)+n(\alpha)}{2} = \gamma$ or $\ell(\alpha) + n(\alpha) = \gamma$ could be used as a convergence order of the Itô-Taylor approximation of SDE solutions.

- **So, we have to characterize the sets $\mathcal{A}_\gamma$ and $\mathcal{B}_\gamma(\mathcal{A}_\gamma)$.**
- **Two types of convergence orders !!!**

THEOREM

Consider the strong Itô-Taylor approximation $Y^{\Delta} = \left\{Y_t^{\Delta},\ t \in]t_0, T[\right\}$ *with convergence order* $\gamma = 0.5, 1, 1.5, \ldots$. *If*

- $\boldsymbol{E}(|X_{t_0}|^2) < \infty \quad$ *and* $\quad \left(\boldsymbol{E}(|X_{t_0} - Y_{t_0}^{\Delta}|^2)\right)^{\frac{1}{2}} \leqslant K_1 \Delta^{\gamma}$
- $|f_\alpha(t,x) - f_\alpha(t,y)| \leqslant K_2|x-y| \qquad \forall \alpha \in \mathcal{A}_\gamma,\ t \in [t_0, T]$ *and* $x, y \in \mathbb{R}^d$,
- $f_{-\alpha} \in C^{1,2} \quad$ *and* $\quad f_\alpha \in H_\alpha \qquad \forall \alpha \in \mathcal{A}_\gamma \cup \mathcal{B}_\gamma$,
- $|f_\alpha(t,x)| \leqslant K_3(1+|x|) \qquad \forall \alpha \in \mathcal{A}_\gamma \cup \mathcal{B}_\gamma, t \in [t_0, T]$ *and* $x \in \mathbb{R}^d$.

then it holds

$$\boldsymbol{E}\left(|X_T - Y_T^{\Delta}|\right) \leqslant K_4 \Delta^{\gamma}. \tag{12}$$

where $K_i, i = 1, 2, 3, 4.$ *are real pairwise independent positive constants .*

For the proof we refer to Kloeden-Platten p. 361ff)

On the Convergence Theorem Assumptions

The convergence theorem assumes that

- the coefficient functions f_α in the Taylor scheme are uniformly bounded on $\mathbb{R}^d$: i.e., the partial derivatives of appropriately high order of the SDE coefficient functions $a, b^1, \ldots, b^m$ are uniformly bounded on R^d.
- However, this assumption is **not satisfied in many important applications**:

1. **Cox-Ingersoll-Ross models** in finance with $V_t \in [0, 1]$

$$dV_t = \alpha(\beta - V_t)dt + \mu\sqrt{V_t}dW_t$$

2. **Duffing-van der Pol oscillator** with multiplicative noise:

$$\begin{aligned} dX_t^1 &= X_t^2 dt \\ dX_t^2 &= \Big(- X_t^1 + \beta X_t^2 - (X_t^1)^3 - (X^1 t)^2 X^2 t \Big) dt + \sigma X_t^2 dW_t \end{aligned}$$

Proofs in the literature (e.g. in the monographs of G. Milstein)

For $\gamma = 0.5, 1, 1.5, 2, \ldots$ the set of Multi-indices $\mathcal{A}_\gamma$ is called Hierarchical Index-Set if

$$\mathcal{A}_\gamma = \{\alpha \in \mathcal{M} : \ \ell(\alpha) + n(\alpha) \leqslant 2\gamma \text{ or } \ell(\alpha) = n(\alpha) = \gamma + \frac{1}{2}\}.$$

- Remark that the three assumption of a Hierarchical set are satisfied.
- The construction of the corresponding remainder set is given by

$$\mathcal{B}(\mathcal{A}_\gamma) = \{\alpha \in \mathcal{M} \backslash \mathcal{A}_\gamma : -\alpha \in \mathcal{A}_\gamma\}.$$

- (Later) γ will refer the convergence order the Itô-Taylor family schemes.

Examples of Hierarchical and Remainder Index-Sets

Consider a SDE driven by one BM, namely W^1. In other words $F = \{0, 1\}$. Thus, we have

- If $\ell_m = 0$ (i.e $\gamma = 0$) then $\mathcal{A}_0 = \{v\}$ and $\mathcal{B}(A_0) = \{(0), (1)\}$.
- If $\ell_m = 1$ (i.e $\gamma = 0.5$) then $\mathcal{A}_{0.5} = \{v, (0), (1)\}$ and

$$\mathcal{B}(\mathcal{A}_{0.5}) = \{(0, 0), (1, 0), (0, 1), (1, 1)\}.$$

- If $\ell_m = 2$ (i.e $\gamma = 1$) then $\mathcal{A}_1 = \{v, (0), (1), (1, 1)\}$ and

$$\mathcal{B}(\mathcal{A}_1) = \{(0, 0), (1, 0), (0, 1), (0, 1, 1), (1, 1, 1)\}.$$

- If $\ell_m = 3$ (i.e $\gamma = 1.5$) then $\mathcal{A}_{1.5} = \{v, (0), (1), (1, 1), (0, 0), (1, 0), (0, 1), (1, 1, 1)\}$ and $\mathcal{B}(\mathcal{A}_{1.5}) =$
$\{(0, 0, 0), (1, 0, 0), (1, 1, 0), (0, 1, 0), (1, 0, 1), (0, 0, 1), (0, 1, 1), (0, 1, 1, 1), (1, 1, 1, 1)\}$.

$$\mathcal{A}_{0.0} = \{v\}$$

$\gamma = 0.5$ Euler-Maruyama

$$\mathcal{A}_{0.5} = \{v, (0), (1)\}$$

$\gamma = 1.$ Milstein/ Taylor

$$\mathcal{A}_{1.0} = \{v, (0), (1), (1,1)\}$$

$\gamma = 1.5$ Taylor

$$\mathcal{A}_{1.5} = \{v, (0), (1), (1,1), (0,0), (1,0), (0,1), (1,1,1)\}$$

$\gamma = 2$ Taylor

$$\mathcal{A}_{2.0} = \Big\{v, (0), (1), (1,1), (0,0), (1,0), (0,1), (1,1,1),$$
$$(1,1,0), (0,1,1), (1,0,1), (0,1,1), (1,1,1,1)\Big\}$$

The strong Itô-Taylor scheme of order $\gamma = 0.5,\ 1,\ 1.5,\ 2, \ldots$ is given by

$$\boxed{Y_{n+1} = Y_n + \sum_{\alpha \in \mathcal{A}_\gamma \setminus \{v\}} I_\alpha[\, f_\alpha(t_n, Y_n)]_{t_n, t_{n+1}}} \tag{13}$$

for $n = 0, 1, 2, \ldots$, and the initial value $Y_0 = \xi_0$.
$\mathcal{M} = \{v\} \bigcup \{\alpha = (j_1, j_2, \ldots, j_l);\ j_i \in F;\ l \in \{1, 2, \ldots\}\}$.

See: M. Zahri, Barycentric interpolation of interface-solution for solving stochastic partial differential equations on non-overlapping subdomains with additive multi-noises, International Journal of Computer Mathematics, Volume 95 (2018), Issue 4, Pages 645-685

Example of Itô-Taylor scheme order 1.5

Consider the autonomous case $dX_t = a(X_t)dt + b(X_t)dW_t$.
An example of Itô-Taylor scheme for a d dimensional Itô-Process X_t and an index set $\mathcal{M} = \{1, \ldots, m\}$ (i.e m dimensional BM) is given by:

$$Y_{n+1} = Y_n + I_v[a(Y_n)]_{t_n,t_{n+1}} + \sum_{j=1}^{m} I_{(j)}[b^j(Y_n)]_{t_n,t_{n+1}} \tag{14}$$

$$+ \sum_{j_1,j_2=1}^{m} I_{(j_1,j_2)}[L^{j_1} b^{j_2}(Y_n)]_{t_n,t_{n+1}} \tag{15}$$

$$+ \sum_{j=1}^{m} I_{(j,0)}[L^j a(Y_n)]_{t_n,t_{n+1}} + \sum_{j=1}^{m} I_{(0,j)}[L^0 b^j(Y_n)]_{t_n,t_{n+1}} \tag{16}$$

$$+ \sum_{j=1}^{m} I_{(0,0)}[a(Y_n)]_{t_n,t_{n+1}} + \sum_{j,j_1,j_2=1}^{m} I_{(j,j_1,j_2)}[L^j L^{j_1} b^{j_2}(Y_n)]_{t_n,t_{n+1}}. \tag{17}$$

Euler Maruyama (**??**), Milstein (**??**)-(**??**) and Taylor of order 1.5 is represented by (**??**)-(**??**).

$$
\begin{aligned}
I_v[a(X_n)]_{t_n,t_{n+1}} &= a(X_n)\Delta t_n \\
I_{(1)}[b^1(X_n)]_{t_n,t_{n+1}} &= b^1(X_n)\int_{t_n}^{t_{n+1}} dW_t^1 \\
&= b(X_n)\Delta W_{t_n} \\
I_{(1,1)}[L^1 b^1(t_n,X_n)]_{t_n,t_{n+1}} &= b^1\frac{\partial b^1}{\partial x}(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW_s^1 dW_t^1 \\
&= b(X_n)\frac{\partial b}{\partial x}(X_n)\frac{1}{2}\left((\Delta W_{t_n})^2 - \Delta t_n\right) \\
I_{(1,0)}[L^1 a(X_n)]_{t_n,t_{n+1}} &= b^1(X_n)\frac{\partial a}{\partial x}(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW_s^1 dt \\
&= b(X_n)\frac{\partial a}{\partial x}(X_n)\Delta Z_{t_n} \qquad \Delta Z_{t_n} \sim N\left(0, \frac{1}{3}(\Delta t_n)^3\right)
\end{aligned}
$$

Recall $\quad L^0 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \frac{1}{2}b^2\frac{\partial}{\partial x^2}, \qquad L^1 = b\frac{\partial}{\partial x}.$

$$
\begin{aligned}
I_{(0,1)}[L^0 b^1(X_n)]_{t_n,t_{n+1}} &= \left(a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\right)(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdW_t \\
&= \left(a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\right)(X_n)\left(\Delta t_n \Delta W_{t_n} - \Delta Z_{t_n}\right) \\
I_{(0,0)}[L^0 a(X_n)]_{t_n,t_{n+1}} &= L^0 a(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdt \\
&= \left(a\frac{\partial a}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 a}{\partial x^2}\right)(X_n)\frac{1}{2}(\Delta t_n)^2 \\
I_{(1,1,1)}[L^1 L^1 b^1(X_n)]_{t_n,t_{n+1}} &= L^1 L^1 b\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\int_{t_n}^{s} dW_r dW_s dW_t \\
&= \left(b\left(\frac{\partial b}{\partial x}\right)^2 + b^2\frac{\partial^2 b}{\partial x^2}\right)(X_n)\left(\frac{(\Delta W_{t_n})^2}{6} - \frac{\Delta t_n}{2}\right)\Delta W_{t_n}
\end{aligned}
$$

Strong Euler-Maruyama Scheme (1955), $\gamma = 0.5$

Consider the **Geometric Brownian Motion**, which satisfies the SDE

$$\boxed{\begin{aligned} X_0 &= \xi \\ dX_t &= \lambda X_t dt + \mu X_t dW_t \end{aligned}}$$

where λ is called the percentage drift and μ is called the percentage volatility. The exact solution is

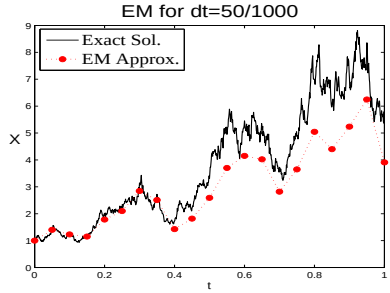
$$X_t = X_0 \exp\left((\lambda - \frac{\mu^2}{2})t - \mu W_t\right)$$

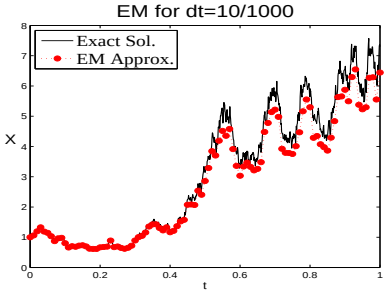
Numerical solution: for $\mathcal{A}_{0.5} = \{v, (0), (1)\}$ (i.e $\ell_1 = 1$), we define the **Euler-Maruyama** scheme

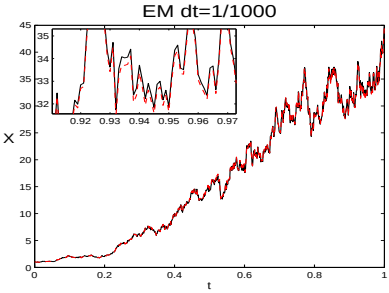
$$\boxed{\begin{aligned} X_0 &= Y_0 = \xi \\ Y_{n+1} &= Y_n + \lambda Y_n \Delta t_n + \mu Y_n \Delta W_{t_n} \end{aligned}}$$

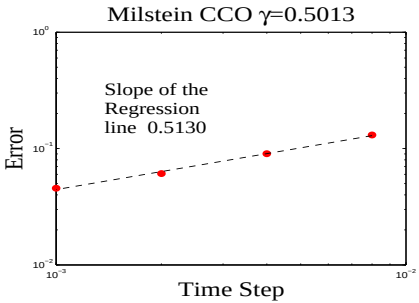
Where Y_n in the numerical approximation of the exact solution X_{t_n}.

Numerical Simulation of GBM by using EM.









Where $dt = 10^{-3}$ is used to simulate exact solution.

Consider the SDE

$$\boxed{\begin{aligned} X_0 &= \xi \\ dX_t &= a(t, X_t)dt + b(t, X_t)dW_t \end{aligned}}$$

Numerical solution: for $\mathcal{A}_{0.5} = \{v, (0), (1)\}$ (i.e $\ell_1 = 1$), we define the **Euler-Maruyama** scheme

$$\boxed{\begin{aligned} X_0 = Y_0 &= \xi \\ Y_{n+1} &= Y_n + a(t_n, Y_n)\Delta t_n + b(t_n, Y_n)\Delta W_{t_n} \end{aligned}}$$

Where Y_n in the numerical approximation of the exact solution X_{t_n} and

$$\begin{aligned} \Delta t_n &= \int_{t_n}^{t_{n+1}} dt \\ \Delta W_{t_n} &= \int_{t_n}^{t_{n+1}} dW_s. \end{aligned}$$

For $\mathcal{A}_1 = \{v, (0), (1), (1,1)\}$ (i.e $\ell_1 = 2$), we define the **Milstein** scheme

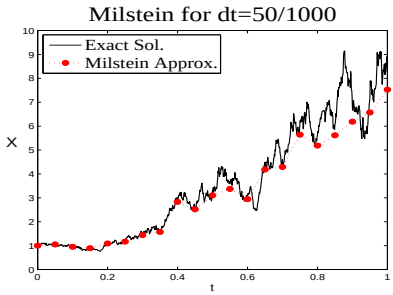
$$\boxed{\begin{aligned} X_0 &= Y_0 = \xi \\ Y_{n+1} &= Y_n + a(t_n, Y_n)\Delta t_n + b(t_n, Y_n)\Delta W_{t_n} \\ &\quad + \frac{1}{2} b(t_n, Y_n)\frac{\partial b}{\partial x}(t_n, Y_n))((\Delta W_{t_n})^2 - \Delta t_n) \end{aligned}} \tag{18}$$

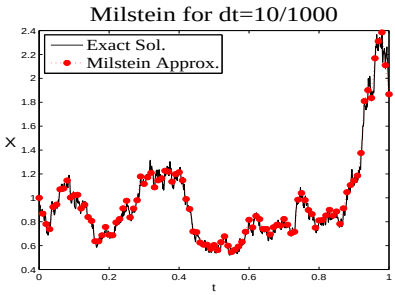
Where Y_n in the numerical approximation of the exact solution X_{t_n} and the additional (to E.M) is

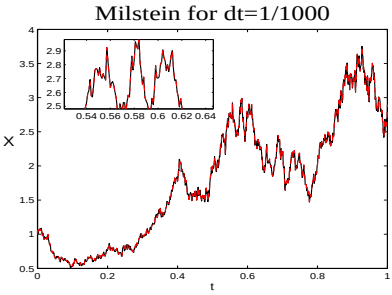
$$f_{(1,1)} I_{(1,1)} \quad = \quad L^1 b(t_n, Y_n) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s^1 dW_t^1 .$$

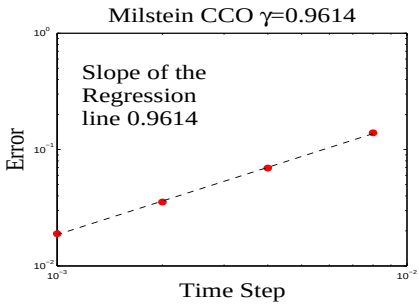
The Milstein method is a first order Itô-Taylor method. It is derived from a truncation of the stochastic Itô-Taylor expansion of the solution.

Numerical Simulation of GBM by using Milstein.









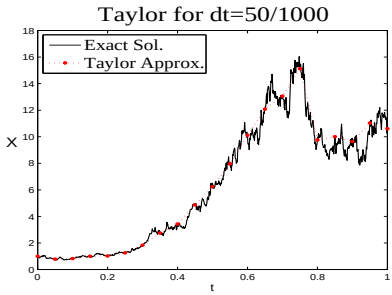
Where $dt = 10^{-3}$ is used to simulate exact solution.

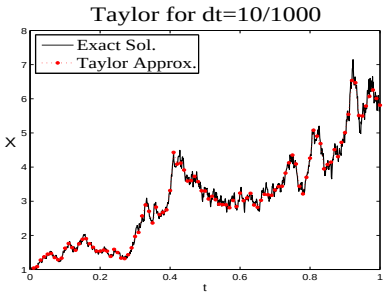
For $\mathcal{A}_{1.5} = \{v, (0), (1), (1,1), (0,0), (1,0), (0,1), (1,1,1)\}$ (i.e $\ell_1 = 3$), we define the **Itô-Taylor** scheme with order $\gamma = 1.5$:

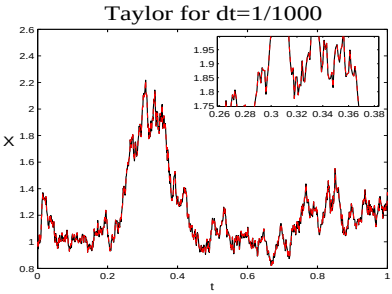
$$
\begin{aligned}
X_0 &= Y_0 = \xi \\
Y_{n+1} &= Y_n + a\Delta t_n + b\Delta W_{t_n} + \frac{1}{2} b \frac{\partial b}{\partial x}((\Delta W_{t_n})^2 - \Delta t_n) \\
&\quad + \frac{1}{2}\left(a\frac{\partial a}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 a}{\partial x^2}\right)(\Delta t)^2 + b\frac{\partial a}{\partial x}\Delta Z_{t_n} \\
&\quad + \left(a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\right)\left(\Delta t_n \Delta W_{t_n} - \Delta Z_{t_n}\right) \\
&\quad + \frac{1}{2}\left(b\left(\frac{\partial b}{\partial x}\right)^2 + b^2\frac{\partial^2 b}{\partial x^2}\right)\left(\frac{1}{3}(\Delta W_{t_n})^2 - \Delta t_n\right)\Delta W_{t_n}
\end{aligned}
\tag{19}
$$

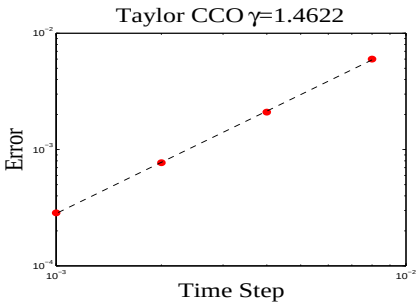
Where Y_n in the numerical approximation of the exact solution X_{t_n} and ΔZ_{t_n} is a normal distributed random variable $\Delta Z_{t_n} \sim N(0, \frac{1}{3}(\Delta t_n)^3)$

Numerical Simulation of GBM by using Taylor.

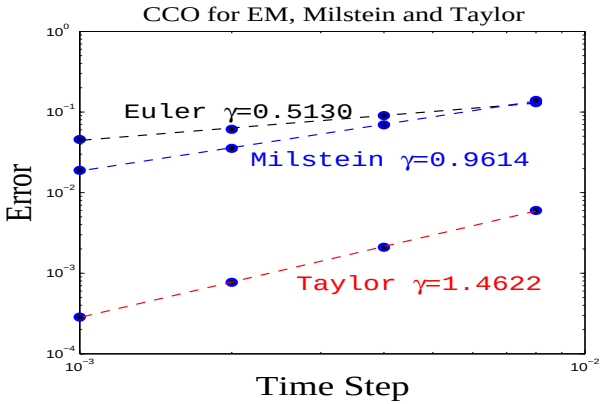








Where $dt = 10^{-3}$ is used to simulate exact solution.



Increasing the number of discretization time-points and the number of simulations leads to the best estimation of computational convergence orders. In our cases, we have used a time step $dt = 10^{-3}$ and $M = 1000$ simulation for each numerical simulation.

Concluding remarks

- Other type of approximations: Weak and Pathwise
- Construction of similar schemes for System of SDEs
- Construction of schemes for SPDEs: time-space dependency
- Study of stability, consistency and scheme properties
- Implicit schemes and relationship to stability
- Multistep stochastic schemes for SDEs and SPDEs
- Extrapolation (Richardson-like) stochastic schemes for SDEs and SPDEs
- Variance Reduction: Reduction of the number of sample for getting same results
- . . .

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THANK YOU

MOSTAFA ZAHRI