# Numerical solution of stochastic partial differential equations (SPDEs) with additive noise on overlapping subdomains

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# Motivation

- Alternating Schwarz method (Schwarz 1870, ASM) and Parallel Schwarz method, (Lions 1988, PSM) for solving deterministic PDEs!

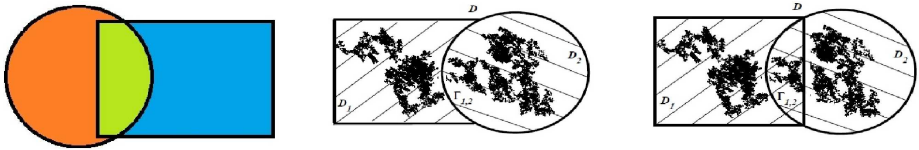


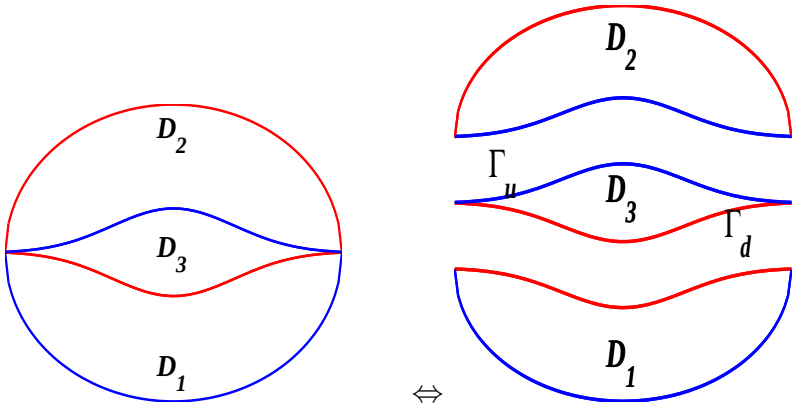

**Figure:** Schwarz overlapped domain decomposition (Fig. 1. left), Non-overlapping sudomains(Fig. 2. middle), Overlapping sudomains(Fig. 3. right)

- Fig.1, ASM: The solution in $D_1$ is a Boundary condition of the Sol. in $D_2$ and vice versa $\Rightarrow$ weak for stochastic partial differential equations (SPDEs).
- Fig.2, PSM: The noise separation of the non-overlapped subdomains leads to the construction of higher order method for approximating SPDEs.
- Fig.3, PSM: The changes in the overlapped region is not an alternating process like in the deterministic case? How to deal with the new function!

# Motivation, cont.

## Overlapped vs Non-overlapped computational domains

**Equivalent problems! Interface solution!**




**(a)** Initial overlapped dom. **(b)** Non-overlapped dom.

Could we find a way to solve the problem on non-overlapped subdomains (right fig.) and to deduce the solution on the overlapped one (left fig.)!

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5. *Itô-Taylor family schemes*
6. *Numerical experiments*
7. *Concluding remarks and outlooks*


This work is an extension of

- M. Zahri, Barycentric interpolation of interface-solution for solving stochastic partial differential equations on non-overlapping subdomains with additive multi-noises, International Jour. of Comp. Math., March 2017.
- M. Zahri, THIS WORK, will appear in Mathematics and Computers in Simulation.

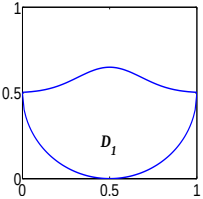
# Modeling of SPDS on overlapping subdomains

## Initial Problem

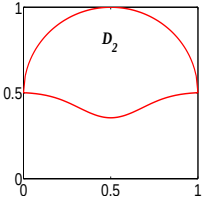
- Overlapping subdomains $\boldsymbol{D}_1$ and $\boldsymbol{D}_2 \subset I\!R^2$: $\boldsymbol{D}_1 \cap \boldsymbol{D}_2 = \Gamma_{1,2}$,
- Stochastic process $\boldsymbol{U}^s : [0,T] \times \boldsymbol{D}_s \times \Omega \longrightarrow I\!R$, $s = 1, 2$
- $(\boldsymbol{U}^1, \boldsymbol{U}^2)$ the exact solution of SPDS:

$$\begin{cases} \boldsymbol{U}^1_t = \nabla \cdot (d_1 \nabla \boldsymbol{U}^1) - a_1 \cdot \nabla \boldsymbol{U}^1 + \sigma^1 \eta^1, & \text{on} \quad \boldsymbol{D}_1, \\ \boldsymbol{U}^2_t = \nabla \cdot (d_2 \nabla \boldsymbol{U}^2) - a_2 \cdot \nabla \boldsymbol{U}^2 + \sigma^2 \eta^2, & \text{on} \quad \boldsymbol{D}_2, \end{cases} \tag{1}$$

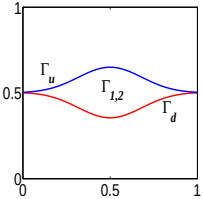
where for $s = 1, 2$, $\eta^s$, are non-necessary correlated Gaussian white noises and $a_s, d_s$ are advection and diffusion coefficients, $\sigma^s$ standard deviation.



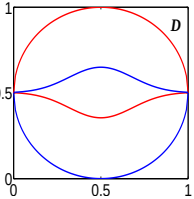

**(c)** Subdom. $D_1$




**(d)** Subdom. $D_2$




**(e)** Interface $\Gamma_{1,2}$




**(f)** Domain $D$

# Non-overlapping equivalent problem

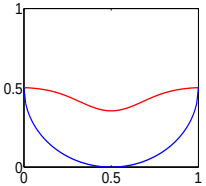
**The Problem**:

- How to approximate $\boldsymbol{U}^1$ and $\boldsymbol{U}^2$ on the **overlapping interface** $\Gamma_{1,2}$ and the **non-overlapping interface** solution on $\partial\Gamma_{1,2} = \Gamma_u \cup \Gamma_d$ ?
- How to deal with **reactions and collisions**... of $\boldsymbol{U}^1$ and $\boldsymbol{U}^2$ on $\Gamma_{1,2}$ ?

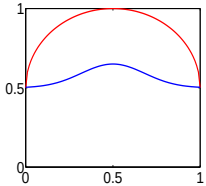
**The Goal**:

- Solving $\boldsymbol{U}^1$ on $\boldsymbol{D}_1 \setminus \Gamma_{1,2}$, solving $\boldsymbol{U}^2$ on $\boldsymbol{D}_2 \setminus \Gamma_{1,2}$ and
- solving $\boldsymbol{U}_t^3 = d_3 \Delta \boldsymbol{U}^3 - a_3 \nabla \boldsymbol{U}^3 + \alpha_t \sigma^1 \eta^1 + \beta_t \sigma^2 \eta^2$, on $\Gamma_{1,2}$,

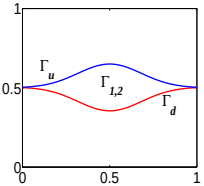
where $\boldsymbol{U}^3 = \alpha_t \boldsymbol{U}^1 + \beta_t \boldsymbol{U}^2$ on $\Gamma_{1,2}$ and the RVs $\alpha_t(\omega), \beta_t(\omega) \in [0, 1]$.




**(a)** $D_1 \setminus \Gamma_{1,2}$




**(b)** $D_2 \setminus \Gamma_{1,2}$




**(c)** Interface $\Gamma_{1,2}$

# Non-overlapping equivalent problem, cont.

The Overlapping problem is transformed into a non-overlapping one:

$$\begin{cases} \boldsymbol{U}_t^1 &= d_1\Delta\boldsymbol{U}^1 - a_1\nabla\boldsymbol{U}^1 + \sigma^1\eta^1 & \text{on} \quad \boldsymbol{D}_1 \setminus \Gamma_{1,2}, \\ \boldsymbol{U}_t^2 &= d_2\Delta\boldsymbol{U}^2 - a_2\nabla\boldsymbol{U}^2 + \sigma^2\eta^2 & \text{on} \quad \boldsymbol{D}_2 \setminus \Gamma_{1,2}, \\ \boldsymbol{U}_t^3 &= d_3\Delta\boldsymbol{U}^3 - a_3\nabla\boldsymbol{U}^3 + \sigma^1\alpha_t\eta^1 + \sigma^2\beta_t\eta^2 & \text{on} \quad \Gamma_{1,2} \\ \mathbf{w}^{1,3} &= \boldsymbol{U}_t^3 = \boldsymbol{U}_t^1 \quad \text{on} \quad \Gamma_d \quad \mathbf{w}^{2,3} = \boldsymbol{U}_t^3 = \boldsymbol{U}_t^2 & \text{on} \quad \Gamma_u \end{cases} \tag{2}$$

where for $s = 1, 2$, $\delta_{\boldsymbol{D}_s}(\mathbf{x}) = 1$, if $\mathbf{x} \in D_s$ and zero otherwise.

- How to approximate $\mathbf{w}^{1,3}$ and $\mathbf{w}^{2,3}$ ? to obtain
- Equivalent Problem: Solving $\boldsymbol{U}$ on $\boldsymbol{D}$, for the governing equation

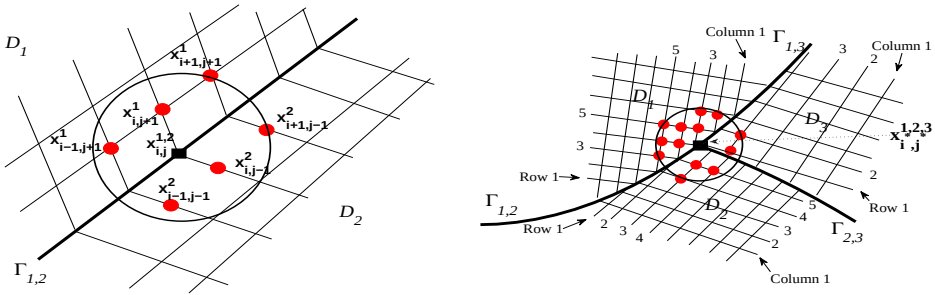
$$\boxed{\boldsymbol{U}_t = d\Delta\boldsymbol{U} - a\nabla\boldsymbol{U} + \delta_{\boldsymbol{D}_1}(\mathbf{x})\sigma^1\eta^1 + \delta_{\boldsymbol{D}_2}(\mathbf{x})\sigma^2\eta^2,} \tag{3}$$

- such that $\boldsymbol{U}^s = \boldsymbol{U} \mid_{\boldsymbol{D}_s \setminus \partial\boldsymbol{D}_s}$ for $s = 1, 2, 3$ and $\boldsymbol{U}^s = \mathbf{w} \mid_{\partial\Gamma_{1,2}}$.

**Compare (2) with (3) is an interface-problem!**

# Barycentric interpolation method (BIM)

## Graphical illustration




**(a)** Point on Linear-interface **(b)** Point on Non-linear-interface

**Figure:** Graphical illustration of the BIM method.

**Which neighbors grid points are candidates for the BIM method !**

# Barycentric interpolation method (BIM)

**Neighbors grid-points set $N_{i,j}^m$ and coefficients set $C_{i,j}$**

For $m \in I\!N$, define the **neighbors grid-points** index set $N_{i,j}^m$ as

$$N_{i,j}^m := \Big\{ (i+a, j+b) \Big| |a| = |b| = m; a, b \in \mathbb{Z} \Big\}, \tag{4}$$

The **coefficients set** $\alpha_{i^*,j^*} \in C_{i,j} \subset ]0,1[$, are

$$\begin{aligned} C_{i,j} \quad := \quad & \Big\{ \Big(\frac{1-2\alpha}{4}\Big)_{i+2,j-2}, \Big(\frac{1-2\alpha}{4}\Big)_{i+2,j+2}, \Big(\frac{\alpha}{2}\Big)_{i+1,j-1}, \Big(\frac{\alpha}{2}\beta^-\Big)_{i+1,j+1}, \\ & \Big(\frac{1-2\alpha}{4}\Big)_{i-2,j-2}, \Big(\frac{1-2\alpha}{4}\Big)_{i-2,j+2}, \Big(\frac{\alpha}{2}\beta^+\Big)_{i-1,j-1}, \Big(\frac{\alpha}{2}\Big)_{i-1,j+1} \Big\}. \end{aligned}$$

and satisfies the following convex combination

$$\boxed{\sum_{\alpha_{i^*,j^*} \in C_{i,j}} \alpha_{i^*,j^*} = \sum_{(i^*,j^*) \in N_{i,j}^1 \cup N_{i,j}^2} \alpha_{i^*,j^*} = 1.} \tag{5}$$

# Barycentric interpolation method

## Domain Decomposition Solution $\boldsymbol{U}^{DD}$

Under appropriate conditions on IS $\boldsymbol{U}^0$ in $L^2(\boldsymbol{D})$, there exist a unique solution $\boldsymbol{U}^{DD}$ in $C(\mathbb{T}, L^2(\boldsymbol{D})) \cap L^2(0,T; H_0^1(\boldsymbol{D})$, such that

$$\boldsymbol{U}^{DD}(\mathbf{x},t) \;:=\; \mathbf{w}(\mathbf{x},t)\big|_{\partial\Gamma_{1,2}} + \sum_{s=1}^{3} \boldsymbol{U}^s(\mathbf{x},t)\big|_{\boldsymbol{D}_s \setminus \partial\boldsymbol{D}_s}, \tag{6}$$

where $\boldsymbol{U}^s$ is the solution of the sub-problem $s$ for $s = 1, 2, 3$.
The non-overlapping interface solution is

$$\mathbf{w}(\mathbf{x}_{i,j},0) \;=\; \boldsymbol{U}^{DD}(\mathbf{x}_{i,j},0) = \boldsymbol{U}^s(\mathbf{x}_{i,j},0),$$

and for $t > 0$, we have

$$\boxed{\mathbf{w}(\mathbf{x}_{i,j},t) = \sum_{(i^*,j^*)\in N^1_{i,j}\cup N^2_{i,j}} \alpha_{i^*,j^*} \boldsymbol{U}^{DD}(\mathbf{x}_{i^*,j^*}, t-\Delta t),} \tag{7}$$

The initial and Dirichlet boundary conditions are

$$\begin{array}{lll} \boldsymbol{U}^s(\mathbf{x},0) & = \boldsymbol{U}(\mathbf{x},0)\big|_{\boldsymbol{D}_s} & \text{for} \quad s = 1,2,3. \\ \boldsymbol{U}^s(\mathbf{x},t) & = 0 & \text{for} \quad (\mathbf{x},t) \in \partial\boldsymbol{D} \times \mathbb{T}. \end{array} \tag{8}$$

# Barycentric interpolation method

## Convergence theorem

- Consider the deterministic case, i.e $\sigma^1 = \sigma^2 = 0$.
- Solving the system (2) on non-overlapping subdomains.
- For any interface point $\mathbf{x}_{i,j} \in \Gamma_u$ or $\Gamma_d$, we assume the following

$$d_3 := \frac{1}{2}(d_1 + d_2) \quad \text{and} \quad a_3 := \frac{1}{2}(a_1 + a_2).$$

**Theorem**

*For $\mathbf{x}_{i,j} \in \Gamma_u$ of $\Gamma_u$, if $4d_3\Delta t < (\Delta x)^2$, $\Delta x < 2d_3/a_3$, then there exist a sequence $(\alpha_{i,j})_k$ such that following two step Barycentric scheme*

$$\boldsymbol{U}(\mathbf{x}_{i,j}, t_{n+1}) \approx U_{i,j}^{n+1} = \sum_{(i^*,j^*)\in N_{i,j}^1 \cup N_{i,j}^2} \alpha_{i^*,j^*} \boldsymbol{U}^{DD}(\mathbf{x}_{i^*,j^*}, t_n), \tag{9}$$

*approximate $U_{i,j}^{n+1}$ in the problem (1) and in the system (2) with time-space convergence order $\mathcal{O}(\Delta t + (\Delta x)^2)$.*

# DDM-MOL-BIM descretization of the SPDS

- Finite differences and the: forward in time centred in space (FTCS),
- Constant temporal step $\Delta t$. Ansatz $\Delta x = \Delta y$.
- The method of lines (MOL): System (2) $\Rightarrow$ Semi-linear system of systems:

$$\begin{aligned} dU^s_{i,j}(t) &= \Big[(-2\alpha)U^s_{i,j}(t) + \frac{\alpha}{2}\beta^- U^s_{i-1,j}(t) + \frac{\alpha}{2}\beta^+ U^s_{i+1,j}(t) \\ &\quad + \frac{\alpha}{2}\beta^- U^s_{i,j-1}(t) + \frac{\alpha}{2}\beta^+ U^s_{i,j+1}\Big]dt + \mathbf{G}^s(t)d\mathbf{W}^s_t \end{aligned} \tag{10}$$

where $s = 1, 2, 3$ and

- $\mathbf{G}^s(t)$ Diffusion matrix, $d\mathbf{W}^s$ Brownian motion vector,
- $\alpha = \dfrac{2d\Delta t}{(\Delta x)^2}$, $\beta^- = 1 - \dfrac{a}{2d}\Delta x$, and $\beta^+ = 1 + \dfrac{a}{2d}\Delta x$.

**The resulting system (10) is a time-problem!**

# DDM-MOL descretization of the SPDS

## Stochastic partial differential system (Itô sense)

In the framework of numerical stochastics: From (10) we obtain

$$\begin{cases} d\boldsymbol{U}^1(t) &= \mathbf{K}^1(t,\boldsymbol{U}^1(t))dt + \mathbf{G}^1(t)d\mathbf{W}^1(t) \quad \text{on} \quad \boldsymbol{D}_1 \setminus \Gamma_{1,2}, \\ d\boldsymbol{U}^2(t) &= \mathbf{K}^2(t,\boldsymbol{U}^2(t))dt + \mathbf{G}^2(t)d\mathbf{W}^2(t) \quad \text{on} \quad \boldsymbol{D}_2 \setminus \Gamma_{1,2}, \\ d\boldsymbol{U}^3(t) &= \mathbf{K}^3(t,\boldsymbol{U}^3(t))dt + \mathbf{G}^3(t)d\mathbf{W}^3(t) \quad \text{on} \quad \Gamma_{1,2} \\ \mathbf{w}^{1,3} &= \boldsymbol{U}_t^3 = \boldsymbol{U}_t^1 \quad \text{on} \quad \Gamma_d \quad \mathbf{w}^{2,3} = \boldsymbol{U}_t^3 = \boldsymbol{U}_t^2 \text{ on } \quad \Gamma_u \end{cases}$$

Which is equivalent to

$$d\boldsymbol{U}(t) = \mathbf{K}(t,\boldsymbol{U}(t))dt + \mathbf{G}(t)d\mathbf{W}(t) \quad \text{on} \quad \boldsymbol{D},$$

where

- $\mathbf{K}^s(t,\boldsymbol{U}^s(t))$ is the drift vector, for $s=1,2,3$, $\Rightarrow \mathbf{K}^s = \mathbf{K}\mid_{\boldsymbol{D}_s \setminus \partial \boldsymbol{D}_s}$ .
- $\mathbf{G}^s(t)$ is the diffusion matrix, for $s=1,2,3$, $\Rightarrow \mathbf{G}^s = \mathbf{G}\mid_{\boldsymbol{D}_s \setminus \partial \boldsymbol{D}_s}$ .

# Itô-Taylor family schemes

For $s = 1, 2, 3$ and $U_0^s = \xi_0$, and $U_n^s$ approximation of $\boldsymbol{U}^s(\cdot, t_n)$ for a given time discretization $\Delta t = t_{n+1} - t_n$. The Itô-Taylor for the system (2) is...

$$
\begin{aligned}
U_{n+1}^s &= Y_n^s + I_{(0)}[\mathbf{K}_s^i(t, U_n^s)]_{\Delta t_n} + \sum_{j_1 \in F_m^1} I_{(j_1)}[\mathbf{G}_{j_1}^s(t, U_n^s)]_{\Delta t_n} && (11)\\
&+ \sum_{(j_1,j_2) \in F_m^2} I_{(j_1,j_2)}[L^{j_1}\mathbf{G}_{j_2}^s(t, U_n^s)]_{\Delta t_n} && (12)\\
&+ \sum_{(j_1,0) \in F_m^1 \times \{0\}} I_{(j_1,0)}[L^{j_1}\mathbf{K}_i^s(t, U_n^s)]_{\Delta t_n}\\
&+ \sum_{(0,j_2) \in \{0\} \times F_m^1} I_{(0,j_2)}[L^0\mathbf{G}_{j_2}^s(t, U_n^s)]_{\Delta t_n}\\
&+ I_{(0,0)}[L^0\mathbf{K}_i^s(t, U_n^s)]_{\Delta t_n} && (13)
\end{aligned}
$$

- Euler Maruyama scheme of order 0.5 $\Rightarrow$ (11),
- Milstein (or Taylor order one) scheme $\Rightarrow$ (11)-(12),
- Taylor scheme of order 1.5 $\Rightarrow$ (11)-(13),

# Itô-Taylor family schemes, cont.

$$
\begin{aligned}
Y_{n+1}^{s} \;=\; & \ldots + \sum_{(j_1,j_2,j_3)\in F_m^3} I_{(j_1,j_2,j_3)}[L^{j_1}L^{j_2}\mathbf{G}_{j_3}^{s}(t,U_n^s)]_{\Delta t_n} \\
& + \sum_{(j_1,j_2,0)\in F_m^2\times\{0\}} I_{(j_1,j_2,0)}[L^{j_1}L^{j_2}\mathbf{K}_{i}^{s}(t,U_n^s)]_{\Delta t_n} \\
& + \sum_{(j_1,0,j_2)\in F_m\times\{0\}\times F_m} I_{(j_1,0,j_3)}[L^{j_1}L^{0}\mathbf{G}_{j_3}^{s}(t,U_n^s)]_{\Delta t_n} \\
& + \sum_{(0,j_2,j_3)\in\{0\}\times F_m^2} I_{(0,j_2,j_3)}[L^{0}L^{j_2}\mathbf{G}_{j_3}^{s}(t,U_n^s)]_{\Delta t_n} \\
& + \sum_{(j_1,j_2,j_3,j_4)\in F_m^4} I_{(j_1,j_2,j_3,j_4)}[L^{j_1}L^{j_2}L^{j_3}\mathbf{G}_{j_4}^{s}(t,U_n^s)]_{\Delta t_n}, \qquad (14)
\end{aligned}
$$

- Taylor scheme of order 2.0 $\Rightarrow$ (11)-(14)
- $L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} a_t^k \frac{\partial}{\partial x^k} + \frac{1}{2}\sum_{k,i=1}^{d}\sum_{j=1}^{m} b^{i,j}b^{k,j}\frac{\partial^2}{\partial x^i \partial x^k}$,
- $L^j = \sum_{i=1}^{d} b^{i,j}\frac{\partial}{\partial x^i}$ and $F_m = \{1,2,\ldots,m\}$ represents the noise index-set.

# Itô-Taylor family schemes

**Stochastic multiple integral** $I_{(j_1,j_2,\ldots)}$

For $\alpha = (j_1, \ldots, j_l)$ and $f \in H_{(j_i)}$ (the set of almost sure square integrable functions), the multiple Itô-integrals are defined per recursion as follows:

$$I_\alpha[f(.)]_{t_0,t} := \begin{cases} f(t), & \text{if } l = 0, \\ \int_{t_0}^t I_{(j_1,\ldots,j_{l-1})}[f(.)]_{t_0,s} ds, & \text{if } l \geq 1 \text{ and } j_l = 0, \\ \int_{t_0}^t I_{(j_1,\ldots,j_{l-1})}[f(.)]_{t_0,s} dW_s^{j_l}, & \text{if } l \geq 1 \text{ and } j_l \geq 1. \end{cases}$$

Here $H_{(\alpha)} := \{f \in H : \ I_{((j_1,\ldots,j_{l-1}))}[f(.)]_{t_0,.} \in H_{(j_l)}\}$. for example:

$$\begin{aligned} I_{(1,2)}[f(.)]_{t_0,t} &= \int_{t_0}^t \int_{t_0}^s f(z) dW_z^1 dW_s^2. \\ I_{(1,2,0)}[f(.)]_{t_0,t} &= \int_{t_0}^t I_{(1,2)}[f(.)]_{0,s} ds = \int_{t_0}^t \int_{t_0}^s \int_{t_0}^{s_1} f(s_2) dW_{s_2}^1 dW_{s_1}^2 ds. \\ I_{(1,2,0,1)}[f(.)]_{t_0,t} &= \int_{t_0}^t I_{(1,2,0)}[f(.)]_{0,s} dW_s^1. \end{aligned}$$

For simplicity, we use the following notations $I_{\alpha,t} = I_\alpha[1]_{0,t}$ and $W_t^0 = t$.

- The process $Y_n$ is a **strong approximation** of $\boldsymbol{U}$ with order $\gamma = 0.5, 1, 1.5, \ldots$, if it exists a constant $K_{p,T} > 0$ such that

$$E_s^{\gamma}(Y_n) := \Big(E|Y_T^{\gamma} - \boldsymbol{U}_T|^p\Big)^{\frac{1}{p}} \leq K_{p,T}(\Delta t)^{\gamma}, \tag{15}$$

- The process $Y_n$ is a **weak approximation** of $\boldsymbol{U}$ with order $\beta = 1, 2, 3, \ldots$, if it exists a constant $K_{g,T} > 0$ such that

$$E_w^{\beta}(Y_n) := \Big|Eg\Big(Y_T^{\beta}\Big) - Eg(\boldsymbol{U}_T)\Big| \leq K_{g,T}(\Delta t)^{\beta}, \tag{16}$$

  where $g$ any polynomial function, $\Delta t$ fixed time step and $p = 1$ od $2$.

- (15)-(16) $\Rightarrow \lim\limits_{\Delta t \to 0} E_s^{\gamma}(Y_n) = \lim\limits_{\Delta t \to 0} E_w^{\beta}(Y_n) = 0.$

- The numerical approximations above are given in $\mathbb{L}^2$.

# Numerical experiments

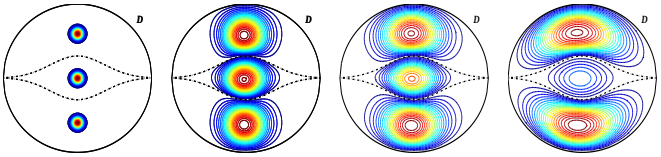
**TEST 1.** **Decoupling vs coupling**: Diffusing local I.S., solving $\boldsymbol{U}^1 \mid_{\boldsymbol{D}_1 \setminus \Gamma_{1,2}}$, $\boldsymbol{U}^2 \mid_{\boldsymbol{D}_2 \setminus \Gamma_{1,2}}$, and $\boldsymbol{U}^3 \mid_{\Gamma_{1,2}}$, ie $\mathbf{w} \equiv 0$ vs $\mathbf{w} \neq 0$. Does $\mathbf{w} \equiv 0$ leads to the decoupling of local solutions?.

**TEST 2.** **Interface smoothing**: Diffusing local uniform distributed I.S., solving $\boldsymbol{U}^1 \mid_{\boldsymbol{D}_1 \setminus \Gamma_{1,2}}$, $\boldsymbol{U}^2 \mid_{\boldsymbol{D}_2 \setminus \Gamma_{1,2}}$, and $(\boldsymbol{U}^1 + \boldsymbol{U}^2) \mid_{\Gamma_{1,2}}$, does $\mathbf{w} \neq 0$ leads to the smoothing of interface solutions?.

**TEST 3.** **Convergence accuracy**: Diffusing normal-like I.S., solving $\boldsymbol{U}^1 \mid_{\boldsymbol{D}_1 \setminus \Gamma_{1,2}}$, $\boldsymbol{U}^2 \mid_{\boldsymbol{D}_2 \setminus \Gamma_{1,2}}$, and $\frac{1}{2}(\boldsymbol{U}^1 + \boldsymbol{U}^2) \mid_{\Gamma_{1,2}}$, for $\mathbf{w} \neq 0$, is the method accurate? Comparing $\boldsymbol{U}^{DD}$ with $\boldsymbol{U}$?

**TEST 4.** **Solving overlapping SPDS**: Diffusing uniform distributed I.S., solving SPDE with additive noise $\boldsymbol{U}^1 \mid_{\boldsymbol{D}_1 \setminus \Gamma_{1,2}}$, $\boldsymbol{U}^2 \mid_{\boldsymbol{D}_2 \setminus \Gamma_{1,2}}$, and $(\alpha_t \boldsymbol{U}^1 + (1 - \alpha_t)\boldsymbol{U}^2) \mid_{\Gamma_{1,2}}$, for $\alpha_t \sim UR(0, 1)$ and $\mathbf{w} \neq 0$. Solution !!! Mean Solution !!!

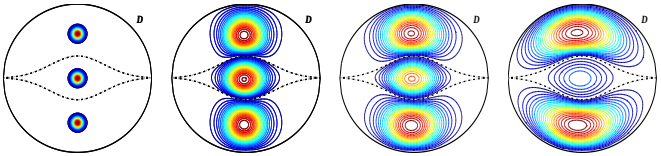
# TEST 1: Decoupling vs coupling of local solutions




**(a)** $t = 0$ **(b)** $t = 100\Delta t$ **(c)** $t = 200\Delta t$ **(d)** $t = 500\Delta t$

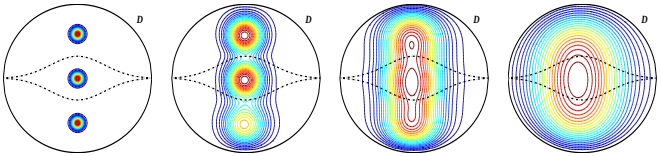
**Figure:** Decoupling: Diffusion on disjunct subdomains $\boldsymbol{D}_1 \setminus \Gamma_{1,2}$, $\boldsymbol{D}_2 \setminus \Gamma_{1,2}$, $\Gamma_{1,2}$.

# TEST 1: Decoupling vs coupling of local solutions




**(a)** $t = 0$ **(b)** $t = 100\Delta t$ **(c)** $t = 200\Delta t$ **(d)** $t = 500\Delta t$

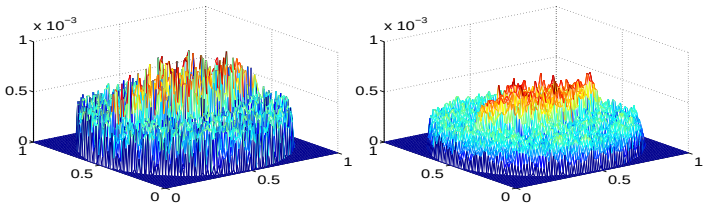
**Figure:** Decoupling: Diffusion on disjunct subdomains $\boldsymbol{D}_1 \setminus \Gamma_{1,2}$, $\boldsymbol{D}_2 \setminus \Gamma_{1,2}$, $\Gamma_{1,2}$.




**(a)** $t = 0$ **(b)** $t = 100\Delta t$ **(c)** $t = 200\Delta t$ **(d)** $t = 500\Delta t$

**Figure:** Coupling: Diffusion using the BIM-Algorithm.

# TEST 2: Interface smoothing




**(a)** $t = 0$ **(b)** $t = 200\Delta t$

**(c)** $t = 300\Delta t$ **(d)** $t = 500\Delta t$

**Figure:** Interface smoothing: Sol. $\boldsymbol{U}^1$ on $\boldsymbol{D}_1 \setminus \Gamma_{1,2}$, $\boldsymbol{U}^2$ on $\boldsymbol{D}_2 \setminus \Gamma_{1,2}$ and $\boldsymbol{U}^3$ on $\Gamma_{1,2}$.

# TEST 3: Convergence accuracy

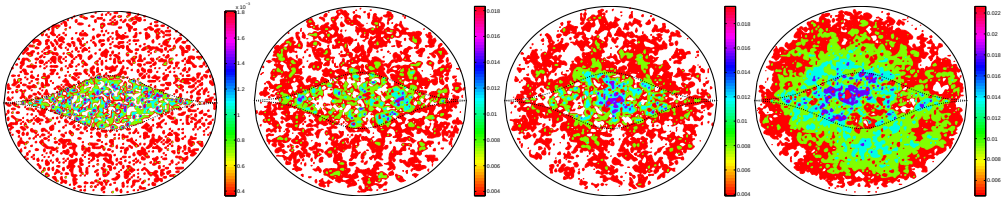
- For $d_1 = d_2 = 0.25$ and $a_1 = a_2 = 0.2 \Rightarrow$ A,
- For $d_1 = 0.25$, $a_1 = 0.2$ and $d_2 = 0.2$, $a_2 = 0.25 \Rightarrow$ B,
- Initial solution $\boldsymbol{U}(\mathbf{x}, 0) = \exp(-30((x-0.5)^2 + (y-0.5)^2))$,
- Interface sol. $\boldsymbol{U}^3 = 0.5(\boldsymbol{U}^1 + \boldsymbol{U}^2)$ on $\Gamma_{1,2}$
- Interface equality $\boldsymbol{U}^1 \mid_{\Gamma_d} = \boldsymbol{U}^3 \mid_{\Gamma_d}$ and $\boldsymbol{U}^2 \mid_{\Gamma_u} = \boldsymbol{U}^3 \mid_{\Gamma_u} \Rightarrow$

| | $A$ | | $B$ | |
|---|---|---|---|---|
| $N_x$ | $L^2$-error | CCO | $L^2$-error | CCO |
| 25 | 2.085266E-3 | – | 3.072218E-3 | – |
| 50 | 1.977757E-4 | – | 5.322484E-4 | – |
| 100 | 2.970599E-5 | **3.0666** | 1.243835E-4 | **2.3132** |
| 200 | 4.084231E-6 | **2.7988** | 3.014155E-5 | **2.0711** |
| 400 | 7.294578E-7 | **2.6738** | 7.511688E-6 | **2.0247** |

- Spatial CCO 2.0 is granted. $\Delta t = 0.95(\Delta t)_{CFL}$, where $(\Delta t)_{CFL} = (\Delta x)^2/4max(d_i)$.

# TEST 4: Solving overlapping SPDEs

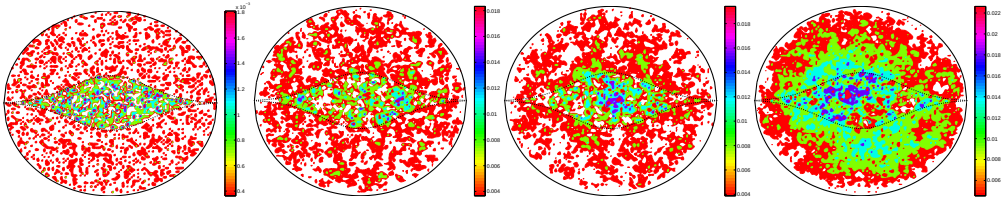
- For $d_1 = d_2 = 0.25$ and $a_1 = a_2 = 0.2 \Rightarrow$ A,
- For $d_1 = 0.25$, $a_1 = 0.2$ and $d_2 = 0.2$, $a_2 = 0.25 \Rightarrow$ B,



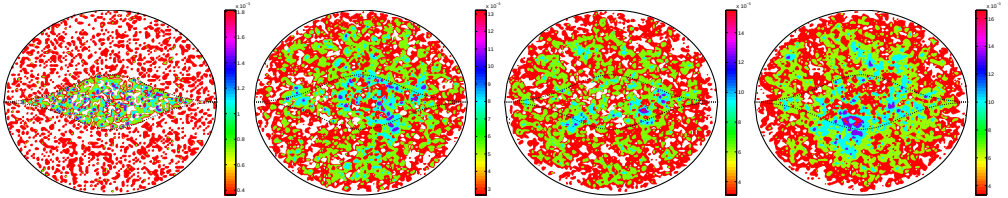
**(a)** A, $t = 0$ **(b)** A, $t = 100\Delta t$ **(c)** A, $t = 200\Delta t$ **(d)** A, $t = 5000\Delta t$

# TEST 4: Solving overlapping SPDEs

- For $d_1 = d_2 = 0.25$ and $a_1 = a_2 = 0.2 \Rightarrow$ A,
- For $d_1 = 0.25$, $a_1 = 0.2$ and $d_2 = 0.2$, $a_2 = 0.25 \Rightarrow$ B,

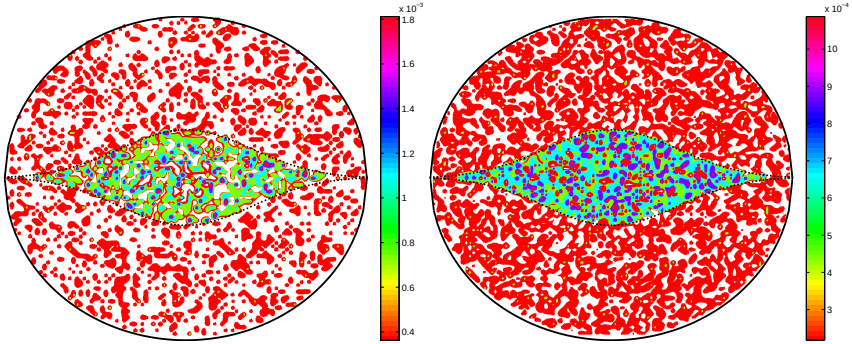


**(a)** A, $t = 0$ **(b)** A, $t = 100\Delta t$ **(c)** A, $t = 200\Delta t$ **(d)** A, $t = 5000\Delta t$



**(e)** B, $t = 0$ **(f)** B, $t = 100\Delta t$ **(g)** B, $t = 200\Delta t$ **(h)** B, $t = 5000\Delta t$

# Initial vs Mean Solution




**(a)** $t = 0$ **(b)** $t = 5000\Delta t, M = 500$

**Figure:** Initial and mean solution at $t = 5000\Delta t$ of $M = 500$ realizations, test A.

# Concluding remarks and outlooks

## Concluding remarks

- Construction of SPDS model on overlapping subdomains.
- Derivation of BIM from finite differences.
- Combination of DDM-MOL-BIM.
- Solving deterministic and stochastic overlapped problems.

## Outlook

- Solving SPDEs on overlapping subdomains using real parameters ...
- Applying the BIM using other schemes: Crank Nicolson, Relaxation ...
- Applying the BIM using other discretization methods: finite volume, finite elements and meshless ...

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**THANK YOU**