Strong Approximation of Stochastic Differential Equations SDEs

Survey and Computational Approach

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Outline

- *Stochastic Differential Equations*
- *Stratonovich and Random Differential Equations*
- *Itô-Formula (Integration with respect to a Brownian Motion)*
- *Itô Taylor-Expansion*
- *Itô Taylor Family Schemes (Strong convergence)*
- *Numerical Examples*

Langevin Equation

Differential equations with noise:

- Langevin used 'noisy' differential equations to model the particle dynamics of Brownian motion in the early 1900s:

$$\boxed{\frac{dx}{dt} = a(t,x) + b(t,x)\eta_t, \qquad \text{white noise} : \eta_t} \tag{1}$$

Gaussian white noise: many mathematical problems finally resolved by Itô in the 1940s with the introduction of stochastic calculus.

- Stochastic differential equations (SDEs)

$$\boxed{\frac{dx}{dt} = a(t,x) + b(t,x)dW_t, \qquad \text{Wiener process} : W_t} \tag{2}$$

The white noise is considered as the differential of the Brownian motion (in distributional sense)

Itô Process, Stochastic differential equation

Let $(W_t)_{t\in\mathbb{T}}$ be an m-dimensional Brownian motion defined on a $(\Omega,\mathcal{A})^m$ with right continuous augmented filtration $\mathcal{F}=(\mathcal{F}_t)_{t\in\mathbb{T}}$. The process $(X_t^1\ldots,X_t^d)$ is called **Itô Process**, iff it has the following form

$$X_t^i = X_{t_0}^i + \underbrace{\int_{t_0}^t a_s^i ds}_{Riemann-integral} + \underbrace{\sum_{j=1}^m \int_{t_0}^t b_s^{i,j} dW_s^j}_{Itô-integral}, \qquad (SDE)$$

- where for all $i=1,\ldots,d$, and $j=1,\ldots,m$; the **drift vector** $(a_t^i)_{t\in\mathbb{T}}$ and the **diffusion matrix** $(b_t^{i,j})_{t\in\mathbb{T}}$ are $\mathcal{F}_t$ adapted and satisfy $\int_{t_0}^T a_s^i ds<\infty$ and $\int_{t_0}^T (b_s^{i,j})^2 ds<\infty$ a.s.
- The (SDE) refers to the **stochastic differential equation** in integral form and in differential form is given by

$$dX_t = a(t,X_t)dt + b(t,X_t)dW_t$$

One dimensional Itô formula (stochastic calculus)

For any twice differentiable scalar function $f(t,x)$ of two real variables t and x, the one dimensional (case $d=m=1$) Itô-Formula is given by

$$\begin{aligned} f(t,X_t) &= f(t_0,X_{t_0}) \\ &+ \int_{t_0}^{t} \left\{ \frac{\partial f}{\partial s}(s,X_s) + a_s \frac{\partial f}{\partial x}(s,X_s) + \frac{1}{2} b_s^2 \frac{\partial^2 f}{\partial x^2}(s,X_s) \right\} ds \\ &+ \int_{t_0}^{t} b_s \frac{\partial f}{\partial x}(s,X_s) dW_s. \end{aligned} \tag{3}$$

Proof idea: By using the Taylor expansion of order two: (all convergence in L^2)

$$\begin{aligned} f(t,X_t) &= f(t_0,X_{t_0}) + \sum_{k=0}^{2^n-1} \Delta f(t_k, X_{t_k}) \\ &= f(t_0,X_{t_0}) + \underbrace{\sum_{k=0}^{2^n-1} \frac{\partial f}{\partial t} \Delta t_k}_{\to \int_{t_0}^{t} \frac{\partial f}{\partial t} ds} + \underbrace{\sum_{k=0}^{2^n-1} \frac{\partial f}{\partial x} \Delta X_{t_k}}_{\to \int_{t_0}^{t} \frac{\partial f}{\partial x} a_s ds + \int_{t_0}^{t} \frac{\partial f}{\partial x} b_s dW_s} + \underbrace{\frac{1}{2} \sum_{k=0}^{2^n-1} \frac{\partial^2 f}{\partial x^2} (\Delta X_{t_k})^2}_{\to \int_{t_0}^{t} \frac{\partial^2 f}{\partial x^2} b_s^2 ds} \\ &+ \sum_{k=0}^{2^n-1} \frac{\partial^2 f}{\partial t \partial x} \Delta t_k \Delta X_{t_k} + \frac{1}{2} \sum_{k=0}^{2^n-1} \frac{\partial^2 f}{\partial t^2} (\Delta t_k)^2 + \sum_{k=0}^{2^n-1} R_k \,. \end{aligned}$$

One dimensional Itô formula: Example and First Result

For $f(t,x) = \frac{1}{2}x^2$ with $X_t = W_t$ and $a_t = 0, b_t = 1$. Applying Itô formula:

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + a_t\frac{\partial f}{\partial x}dt + b_t\frac{\partial f}{\partial x}dW_t + \frac{1}{2}b_t^2\frac{\partial^2 f}{\partial x^2}dt \\ &= \frac{\partial f}{\partial t}dt + (1)\frac{\partial f}{\partial x}dW_t + \frac{1}{2}(1)^2\frac{\partial^2 f}{\partial x^2}dt \\ &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt \\ &= \frac{\partial f}{\partial x}dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt \end{aligned}$$

Hence $\frac{1}{2}dW_t^2 = W_t dW_t + \frac{1}{2}dt$ and $\frac{1}{2}\int dW_s^2 = \int W_s dW_s + \frac{1}{2}\int dt$. Thus,

$$\boxed{I_t = \int_0^t W_s dW_s = \frac{1}{2}(W_t^2 - t)}$$

The stochastic integral with respect to a Brownian motion is different from the Riemanian one $\int_0^x sds = \frac{1}{2}x^2$.

Exact Solution of SDE (Theorem)

Let a (**drift**) and b (**diffusion**) are two measurable vector functions defined as follows: $a : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $b : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$. If the $d-$dim SDE with a F_{t_0}-measurable initial condition $X_{t_0} = \Upsilon$ (RV), independent from $\sigma(W)$, satisfies:

1. (**Lipschitz condition**) For $t \in [t_0, T],\ x, y \in \mathbb{R}^d$ and a constant $K > 0$:

$$\|a(t,x) - a(t,y)\| \leqslant K\|x - y\| \quad \text{and} \quad \|b(t,x) - b(t,y)\| \leqslant K\|x - y\|$$

2. (**Linear Growth**) For $t \in [t_0, T],\ x \in \mathbb{R}^d$ and a constant $D > 0$:

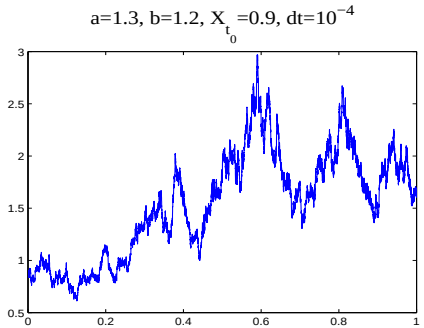
$$\|a(t,x)\|^2 + \|b(t,x)\|^2 \leqslant D(1 + \|x\|^2).$$

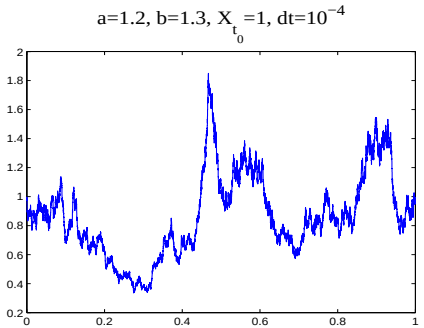
then the SDE has a **pathwise unique solution** X_t on $[t_0, T]$ with $E\left(\sup_{t_0 \leqslant s \leqslant T} |X_s|^2\right) < \infty$.

Simulation of Exact Solution of a SDE

Consider the linear SDE: $X_t = aX_t ds + bX_t dW_t$ with exact solution

$$X_t = X_{t_0} \exp\left((a - \frac{1}{2}b^2)t + bW_t\right)$$





Random and Stochastic ODEs

- Random ordinary differential equations (RODEs) are pathwise ODEs that contain a stochastic process in their vector field functions:

$$x'(t) = f(\xi(\omega), x(t)) \Longleftrightarrow x'_{\omega}(t) = g_{\omega}(t, x(t)) \quad \text{for} \quad \omega \in \Omega \quad (a.s)$$

- Clearly, once ω is fixed, all classical numerical schemes for ODEs are pathwise applicable for RODEs.
- Stochastic ordinary differential equations (SODEs) or (SDEs) are differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is itself a stochastic process.
- Consequently, classical numerical schemes for ODEs are not directly applicable for SDE.

SO, we need stochastic Differentiation and Integration.

Stratonovich and Stochastic ODEs, cont.

Consider the **Stratonovich RODE** with a Wiener process

$$\boxed{dX_t = a(X_t) + b(X_t) \circ dW_t} \tag{4}$$

The Stratonovich SDE can be rewritten as stochastic differential system of equations

Example

$$\frac{dx}{dt} = -x + \sin(W_t) \Leftrightarrow d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} -X_t + \sin(Y_t) \\ 0 \end{pmatrix} dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dW_t$$

So, We can use deterministic calculus pathwise for RODEs, however these methods rarely attain their traditional order, thus new forms of higher order schemes are required.[1]

[1] *Peter Kloeden and Arnuf Jentzen (Proc. R. Soc. A (2007)).*

About the present Talk

- The main concern of this talk is the construction of the Itô-Taylor family scheme of SDEs of type

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t,$$

and to approximate their numerical solutions by using a strong criteria in the L^2-theory.

- Solving systems of type ($i = 1, \ldots, d$ and $j = 1, \ldots, m$)

$$X_t^i = X_{t_0}^i + \int_{t_0}^t a_s^i ds + \sum_{j=1}^m \int_{t_0}^t b_s^{i,j} dW_s^j,$$

requires more details about multiple stochastic integrals. These are not subject of the present talk.

Strong and Weak convergence

Denotes by X_t the exact solution of an SDE and denotes by Y_t the corresponding numerical solution to the SDE for a given time discretization of $[t_0, T]$ such that $t_0 < t_1 < \ldots < t_{N_T} = T$ with a maximal step-size Δ, converges to the e. We have to distinct between the following types of convergence :

- We said that the c**convergence is strong** of order γ if

$$\lim_{N_T \to \infty} E_s(\gamma) := \lim_{N_T \to \infty} E\Big(|Y^{\gamma}_{N_T} - X_T|\Big) = 0.$$

- We said that the **convergence is weak** of order γ if

$$\lim_{N_T \to \infty} E_w(\gamma) := \lim_{N_T \to \infty} \Big|Eg\Big(Y^{\gamma}_{N_T}\Big) - Eg(X_T)\Big| = 0.$$

Where N_T refers the number of the time steps used for the approximation, g any polynomial function.

One-dim Itô coefficient functions

For $Y_t = f(t, X_t)$ consider the one dimensional SDE

$$dY_t = a(t, X_t)dt + b(t, X_t)dW_t$$

The one dimensional Itô Formula is given by

$$\boxed{dY_t = L^0 f(t, X_t)dt + L^1 f(t, X_t)dW_t,} \tag{5}$$

Where the differential operators L^0 and L^1 are given as follows:

$$L^0 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \underbrace{\frac{1}{2}b^2\frac{\partial^2}{\partial x^2}}_{\text{due to } E(\Delta W_t)=\Delta t} \qquad \text{and} \qquad L^1 = b\frac{\partial}{\partial x}.$$

- L^0 and L^1 are called **Itô differential operators**.
- $f_{(0)} := L^0 f$ and $f_{(1)} := L^1$ are called **Itô coefficient functions**.
- in general $f_{(j_1,j_2,\ldots,j_l)} := L^{j_1}L^{j_2}\cdots L^{j_\ell} f$, where $j_i \in \{0, 1\}$ for $i = 1, \ldots, \ell$.

Multi-indices (to generalize the

For $m \in \mathbb{N}$ and $F = \{0, 1, \ldots, m\}$.For $\alpha = (j_1, \ldots, j_l)$ where $j_i \in F$ and $1 \leqslant i \leqslant l$. We denote the **size** and the **number of zero components**of α

$$\begin{aligned} \ell(\alpha) &:= \ell_\alpha \qquad number\ of\ components\ of\ \alpha \\ n(\alpha) &:= n_\alpha \qquad number\ of\ zero\ components\ of\ \alpha \end{aligned}$$

The set of all multi-indices with respect to F is represented by

$$\mathcal{M} = \bigcup_{l=1}^{\infty} F^l \cup \{v\}, \tag{6}$$

where v refers to the empty multi-index with size zero. The following example gives more sense for the definition above:

Example

For $\alpha = (1, 0, 2)$ it holds $\ell(\alpha) = 3$ and $n(\alpha) = 1$.
For $\alpha = (1, 0, 0, 2, 3, 1, 0, 0)$ it holds $(\alpha) = 8$ and $n(\alpha) = 4$.

Operation on Multi-index set

- (The **drop** $-$ operator). For $\alpha \in \mathcal{M}$ with $\alpha = (j_1, j_2, \ldots, j_l)$. For $l \geqslant 1$, we define $\alpha-$ and $-\alpha$ as follows:

$$\begin{aligned} \alpha- &:= (j_1, j_2, \ldots, j_{l-1}) \quad \text{drop last component.} \\ -\alpha &:= (j_2, \ldots, j_l) \quad \text{drop last component.} \end{aligned}$$

  If $\ell(\alpha) = l > 1$ then, it implies $l(-\alpha) = l(\alpha-) = l - 1$.
  If $\ell(\alpha) = l = 1$ then, it implies $-\alpha = \alpha- = v$ and $l(-\alpha) = l(\alpha-) = 0$.

- (The **star** $\star$ operator). Let us consider
  $\alpha = (j_1, j_2, \ldots, j_l), \beta = (i_1, i_2, \ldots, i_k) \in \mathcal{M}$.
  The operator $\star$ is defined as:

$$\begin{aligned} \alpha * \beta &:= (j_1, j_2, \ldots, j_l, i_1, i_2, \ldots, i_k) \quad \text{and} \\ \beta * \alpha &:= (i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_l). \end{aligned}$$

Adaptation of the Stochastic Processes

The stochastic processes, which are solution of SDEs should be at each time step measurable with respect the an appropriate filtration:

1. Therefore, define the set H as a set of stochastic processes $(f_t)_{t\geqslant 0}$, which are progressively adapted to the associated filtration $\{\mathcal{F}_t\}_{t\geqslant 0})$, right continuous and the left limit exists. Moreover, define the sets $H_v, H_{(0)}, H_{(1)}$ as follows
2. $H_v := \{f \in H : \forall\ t \geqslant t_0 \quad |f(t,w)| < \infty \quad \text{a.s.}\}$,
3. $H_{(0)} := \left\{f \in H : \forall\ t \geqslant t_0 \quad \int_{t_0}^{t} |f(s,w)|ds < \infty \quad \text{a.s.}\right\}$,
4. $H_{(1)} := \left\{f \in H : \forall\ t \geqslant t_0 \quad \int_{t_0}^{t} |f(s,w)|^2 ds < \infty \quad \text{a.s.}\right\}$.

For $j \in F\backslash\{0\}$ one sets $H_{(j)} = H_{(1)}$.

Useful Notation for multiple Stochastic Integrals

Let us consider $\alpha = (j_1, j_2, \ldots, j_l)$ a multi-index and $(W_t)_{t \geqslant 0}$ an m-dimensional Brownian motion. For $f \in H_{(\alpha)}$, multiple Itô-Integrals are defined per recursion as follows:

$$I_\alpha[f(.)]_{t_0,t} := \begin{cases} f(t), & \text{if } l = 0 \\ \int_{t_0}^{t} I_{\alpha-}[f(.)]_{t_0,s} ds, & \text{if } l \geqslant 1 \text{ and } j_l = 0 \\ \int_{t_0}^{t} I_{\alpha-}[f(.)]_{t_0,s} dW_s^{j_l}, & \text{if } l \geqslant 1 \text{ and } j_l \geqslant 1. \end{cases}$$

here $H_{(\alpha)}$ is defined per recursion as
$H_{(\alpha)} := \{f \in H : \quad I_{(\alpha-)}[f(.)]_{t_0,.} \in H_{(j_l)}\}$, for $j_l \in F$ and $l \geqslant 2$.

Example

$$\begin{aligned} I_{(1,2)}[f(.)]_{t_0,t} &= \int_{t_0}^{t} \int_{t_0}^{s} f(z) dW_z^1 dW_s^2, \\ I_{(1,2,0)}[f(.)]_{t_0,t} &= \int_{t_0}^{t} I_{(1,2)}[f(.)]_{0,s} ds = \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{s_1} f(s_2) dW_{s_2}^1 dW_{s_1}^2 ds. \end{aligned}$$

One dimensional Itô Taylor Expansion

Consider the one dimensional Itô-Process X_t, which satisfy the Itô-Integral equation:

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)ds + \int_{t_0}^{t} b(X_s)dW_s \tag{7}$$

where a and b reel valued smooth functions, satisfying time-growth condition and $f \in C^2(\mathbb{R}, \mathbb{R})$. By applying the Itô Formula for $f(X_s)$, it follows:

$$\begin{aligned} f(X_s) &= f(X_{t_0}) + \int_{t_0}^{t} \left(a(X_s)\frac{\partial}{\partial x} f(X_s) + \frac{1}{2} b^2(X_s)\frac{\partial^2}{\partial x^2} f(X_s) \right) ds \\ &\quad + \int_{t_0}^{t} b(X_s)\frac{\partial}{\partial x} f(X_s) dW_s \\ &= f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s) ds + \int_{t_0}^{t} L^1 f(X_s) dW_s. \end{aligned}$$

One dimensional Itô Taylor Expansion: Example

If $f(x) = x \in \mathbb{R}$ then $L^0 f = a$ and $L^1 f = b$ the Itô-Taylor Expansion is given in integral form as

$$\begin{aligned} X_t &= X_{t_0} + \int_{t_0}^{t} \left(a(X_{t_0}) + \int_{t_0}^{s} L^0 a(X_z) dz + \int_{t_0}^{s} L^1 a(X_z) dW_z \right) ds \\ &\quad + \int_{t_0}^{t} \left(b(X_{t_0}) + \int_{t_0}^{s} L^0 b(X_z) dz + \int_{t_0}^{s} L^1 b(X_z) dW_z) \right) dW_s \\ &= X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s + \underbrace{R_1}_{Remainder} \end{aligned}$$

where the remainder term is given by:

$$\begin{aligned} R_1 &= \int_{t_0}^{t} \int_{t_0}^{s} L^0 a(X_z) dz ds + \int_{t_0}^{t} \int_{t_0}^{s} L^1 a(X_z) dW_z ds \\ &\quad + \int_{t_0}^{t} \int_{t_0}^{s} L^0 b(X_z) dz dW_s + \int_{t_0}^{t} \int_{t_0}^{s} \underbrace{L^1 b(X_z)}_{Expand\ next} \, dW_z dW_s. \end{aligned}$$

By applying the Itô-Formula to the function $L^1 b$, we get

$$\begin{aligned} X_t &= X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s \\ &\quad + L^1 b(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dW_z dW_s + \underbrace{R_2}_{Remainder} \end{aligned}$$

where the remainder term is given by:

$$\begin{aligned} R_2 &= \int_{t_0}^{t} \int_{t_0}^{s} L^0 a(X_z) dz ds + \int_{t_0}^{t} \int_{t_0}^{s} L^1 a(X_z) dW_z ds \\ &\quad + \int_{t_0}^{t} \int_{t_0}^{s} L^0 b(X_z) dz dW_s + \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^0 L^1 b(X_u) du dW_z dW_s \\ &\quad + \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^1 \underbrace{L^1 b(X_u)}_{Expand\ next} dW_u dW_z dW_s. \end{aligned}$$

Don't forget $dtdt = dtdW_t = dW_t dt = 0$ and $dW_t dW_t = dt$.

Multi-Index: Hierarchical and Remainder Index-Sets

A set $\mathcal{A}$ of Multi-Index is called **Hierarchical Index-Set**, if the following condition are satisfied

- $\mathcal{A} \neq \varnothing$,
- $\ell_m = \sup_{\alpha \in \mathcal{A}} \quad l(\alpha) < \infty$,
- $-\alpha \in \mathcal{A}; \quad \forall\ \alpha \in \mathcal{A} \backslash \{v\}$.

For any hierarchical set $\mathcal{A}$, we define the corresponding **Remainder Index-Set** as:

$$\mathcal{B}(\mathcal{A}) = \{\alpha \in \mathcal{M} \backslash \{\mathcal{A}\} : -\alpha \in \mathcal{A}\}.$$

General Form of the Itô-Taylor-Expansion

Theorem

Let ρ and τ are two Stop-times such that $t_0 \leqslant \rho(w) \leqslant \tau(w) \leqslant T$. If the function $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}$ and all its differential are smooth and if the partial differential of a and b exist, then for any set $\mathcal{A} \in \mathcal{M}$ follow the stochastic Taylor-Expansion

$$f(\tau, X_\tau) = \sum_{\alpha \in \mathcal{A}_\gamma} I_\alpha[f_\alpha(\rho, X_\rho)] + \sum_{\alpha \in \mathcal{B}(\mathcal{A}_\gamma)} I_\alpha[f_\alpha(., X.)]_{\rho,\tau}, \tag{8}$$

Where the **Itô coefficient functions** are defined by

$$f_\alpha = \begin{cases} f & \text{if} \quad \ell(\alpha) = 0; \\ L^{j_1} f_{-\alpha} & \text{if} \quad \ell(\alpha) > 0. \end{cases}$$

for $\alpha = (j_1, \ldots, j_l) \in M$ and $f \in C^{n(\alpha), \ell(\alpha)+n(\alpha)}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R})$.

Convergence Theorem: Source of the convergence order

For $l \in \mathbb{N}$ and $\alpha = (j_1, \ldots, j_l)$ it exists a positive constant K independent from Δ such that

$$\boxed{\left(I\!E(I_{\alpha,\Delta}^2)\right)^{\frac{1}{2}} \leqslant K\Delta^{\frac{\ell(\alpha)+n(\alpha)}{2}}} \tag{9}$$

Moreover, for X a d-dim stoch. process sol. of SDE, a multi-Index $\alpha \in \mathcal{M}$ and $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}$ satisfying If $|f_\alpha(t,x)|^2 \leqslant K\,(1 + |x|^2)$, the we have

$$\boxed{I\!E\left|I_\alpha\left[f_\alpha(t, X_t)\right]_{0,\Delta}\right|^2 \leqslant K\left(1 + I\!E|X_t|^2\right)\Delta^{l(\alpha)+n(\alpha)}} \tag{10}$$

Remark that the real nuber $\frac{\ell(\alpha)+n(\alpha)}{2} = \gamma$ could be used as a convergence order of the Itô-Taylor approximation of SDE solutions.

So, we have to characterize the sets $\mathcal{A}_\gamma$ and $\mathcal{B}_\gamma(\mathcal{A}_\gamma)$.

Convergence Theorem, cont.

Theorem

Consider the strong Itô-Taylor approximation $Y^{\Delta} = \left\{ Y_t^{\Delta},\ t \in]t_0, T[\right\}$ *with convergence order* $\gamma = 0.5, 1, 1.5, \ldots$. *If*

- $I\!E(|X_{t_0}|^2) < \infty$ *and* $\left(I\!E(|X_{t_0} - Y_{t_0}^{\Delta}|^2) \right)^{\frac{1}{2}} \leqslant K_1 \Delta^{\gamma}$
- $|f_\alpha(t,x) - f_\alpha(t,y)| \leqslant K_2 |x-y| \qquad \forall \alpha \in \mathcal{A}_\gamma,\ t \in [t_0, T]$ *and* $x, y \in \mathbb{R}^d$,
- $f_{-\alpha} \in C^{1,2}$ *and* $f_\alpha \in H_\alpha \qquad \forall \alpha \in \mathcal{A}_\gamma \cup \mathcal{B}_\gamma$,
- $|f_\alpha(t,x)| \leqslant K_3(1 + |x|) \qquad \forall \alpha \in \mathcal{A}_\gamma \cup \mathcal{B}_\gamma, t \in [t_0, T]$ *and* $x \in \mathbb{R}^d$.

then it holds

$$I\!E\left(|X_T - Y_T^{\Delta}| \right) \leqslant K_4 \Delta^{\gamma}. \tag{11}$$

where $K_i, i = 1, 2, 3, 4.$ *are real pairwise independent positive constants .*

For the proof we refer to [2] p. 361ff)

On the Convergence Theorem Assumptions

The convergence theorem assumes that

- the coefficient functions f_α in the Taylor scheme are uniformly bounded on $\mathbb{R}^d$: i.e., the partial derivatives of appropriately high order of the SDE coefficient functions $a, b^1, \ldots, b^m$ are uniformly bounded on R^d.
- However, this assumption is **not satisfied in many important applications**:

1. **Cox-Ingersoll-Ross models** in finance with $V_t \in [0, 1]$

$$dV_t = \alpha(\beta - V_t)dt + \mu\sqrt{V_t}dW_t$$

2. **Duffing-van der Pol oscillator** with multiplicative noise:

$$\begin{aligned} dX_t^1 &= X_t^2 dt \\ dX_t^2 &= \Big(- X_t^1 + \beta X_t^2 - (X_t^1)^3 - (X_t^1)^2 X_t^2 \Big) dt + \sigma X_t^2 dW_t \end{aligned}$$

Proofs in the literature (e.g. in the monographs) see [2, 4]

Construction of Multi-Index Sets in the Itô-Taylor-Expansion

For $\gamma = 0.5, 1, 1.5, 2, \dots$ the set of Multi-indices $\mathcal{A}_\gamma$ is called Hierarchical Index-Set if

$$\mathcal{A}_\gamma = \{\alpha \in \mathcal{M} : \ \ell(\alpha) + n(\alpha) \leqslant 2\gamma \ or \ \ell(\alpha) = n(\alpha) = \gamma + \frac{1}{2}\}.$$

- Remark that the three assumption of a Hierarchical set are satisfied.
- The construction of the corresponding remainder set is given by

$$\mathcal{B}(\mathcal{A}_\gamma) = \{\alpha \in \mathcal{M} \backslash \mathcal{A}_\gamma : -\alpha \in \mathcal{A}_\gamma\}.$$

- (Later) γ will refer the convergence order the Itô-Taylor family schemes.

Examples of Hierarchical and Remainder Index-Sets

For $F = \{0, 1\}$, we have

- If $\ell_m = 0$ (i.e $\gamma = 0$) then $\mathcal{A}_0 = \{v\}$ and $\mathcal{B}(A_0) = \{(0), (1)\}$.
- If $\ell_m = 1$ (i.e $\gamma = 0.5$) then $\mathcal{A}_{0.5} = \{v, (0), (1)\}$ and

$$\mathcal{B}(\mathcal{A}_{0.5}) = \{(0,0), (1,0), (0,1), (1,1)\}.$$

- If $\ell_m = 2$ (i.e $\gamma = 1$) then $\mathcal{A}_1 = \{v, (0), (1), (1,1)\}$ and

$$\mathcal{B}(\mathcal{A}_1) = \{(0,0), (1,0), (0,1), (0,1,1), (1,1,1)\}.$$

- If $\ell_m = 3$ (i.e $\gamma = 1.5$) then $\mathcal{A}_{1.5} = \{v, (0), (1), (1,1), (0,0), (1,0), (0,1), (1,1,1)\}$ and $\mathcal{B}(\mathcal{A}_{1.5}) =$
$\{(0,0,0), (1,0,0), (1,1,0), (0,1,0), (1,0,1), (0,0,1), (0,1,1), (0,1,1,1), (1,1,1,1)\}$.

Hierarchical Index-Sets, cont.

$$\mathcal{A}_{0.0} = \{v\}$$

$\gamma = 0.5$ Euler-Maruyama

$$\mathcal{A}_{0.5} = \{v, (0), (1)\}$$

$\gamma = 1.$ Milstein/ Taylor

$$\mathcal{A}_{1.0} = \{v, (0), (1), (1,1)\}$$

$\gamma = 1.5$ Taylor

$$\mathcal{A}_{1.5} = \{v, (0), (1), (1,1), (0,0), (1,0), (0,1), (1,1,1)\}$$

$\gamma = 2$ Taylor

$$\mathcal{A}_{2.0} = \Big\{v, (0), (1), (1,1), (0,0), (1,0), (0,1), (1,1,1),$$
$$(1,1,0), (0,1,1), (1,0,1), (0,1,1), (1,1,1,1)\Big\}$$

Itô-Taylor Family Schemes: Strong Convergence

The Itô-Taylor scheme of strong order $\gamma = 0.5,\ 1,\ 1.5,\ 2, \ldots$ is given by

$$\boxed{Y_{n+1} = Y_n + \sum_{\alpha \in \mathcal{A}_\gamma \setminus \{v\}} I_\alpha[f_\alpha(t_n, Y_n)]_{t_n, t_{n+1}}} \tag{12}$$

for $n = 0, 1, 2, \ldots$, and the initial value $Y_0 = \xi_0$.
$\mathcal{M} = \{v\} \bigcup \{\alpha = (j_1, j_2, \ldots, j_l); j_i \in F;\ l \in \{1, 2, \ldots\}\}$.

Example of Itô-Taylor scheme order 1.5

Consider the autonomous case $dX_t = a(X_t)dt + b(X_t)dW_t$.
An example of Itô-Taylor scheme for a d dimensional Itô-Process X_t and an index set $\mathcal{M} = \{1, \dots, m\}$ (i.e m dimensional BM) is given by:

$$
\begin{aligned}
Y_{n+1} &= Y_n + I_v[a(Y_n)]_{t_n,t_{n+1}} + \sum_{j=1}^{m} I_{(j)}[b^j(Y_n)]_{t_n,t_{n+1}} && (13)\\
&+ \sum_{j_1,j_2=1}^{m} I_{(j_1,j_2)}[L^{j_1}b^{j_2}(Y_n)]_{t_n,t_{n+1}} && (14)\\
&+ \sum_{j=1}^{m} I_{(j,0)}[L^j a(Y_n)]_{t_n,t_{n+1}} + \sum_{j=1}^{m} I_{(0,j)}[L^0 b^j(Y_n)]_{t_n,t_{n+1}} && (15)\\
&+ \sum_{j=1}^{m} I_{(0,0)}[a(Y_n)]_{t_n,t_{n+1}} + \sum_{j,j_1,j_2=1}^{m} I_{(j,j_1,j_2)}[L^j L^{j_1}b^{j_2}(Y_n)]_{t_n,t_{n+1}} && (16)
\end{aligned}
$$

Euler Maruyama (13), Milstein (13)-(14) and Taylor of order 1.5 is represented by (13)-(16).

The one-dimensional Itô-Taylor $\gamma = 1.5$, $(d = m = 1)$

$$
\begin{aligned}
I_v[a(X_n)]_{t_n,t_{n+1}} &= a(X_n)\Delta t_n \\
I_{(1)}[b^1(X_n)]_{t_n,t_{n+1}} &= b^1(X_n)\int_{t_n}^{t_{n+1}} dW_t^1 \\
&= b(X_n)\Delta W_{t_n} \\
I_{(1,1)}[L^1 b^1(t_n,X_n)]_{t_n,t_{n+1}} &= b^1\frac{\partial b^1}{\partial x}(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW_s^1 dW_t^1 \\
&= b(X_n)\frac{\partial b}{\partial x}(X_n)\frac{1}{2}\left((\Delta W_{t_n})^2 - \Delta t_n\right) \\
I_{(1,0)}[L^1 a(X_n)]_{t_n,t_{n+1}} &= b^1(X_n)\frac{\partial a}{\partial x}(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW_s^1 dt \\
&= b(X_n)\frac{\partial a}{\partial x}(X_n)\Delta Z_{t_n} \qquad \Delta Z_{t_n} \sim N\left(0, \frac{1}{3}(\Delta t_n)^3\right)
\end{aligned}
$$

Recall $\quad L^0 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \frac{1}{2}b^2\frac{\partial}{\partial x^2}, \qquad L^1 = b\frac{\partial}{\partial x}.$

The 1-dim case Itô-Taylor $\gamma = 1.5$, $(d = m = 1)$ cont.

$$
\begin{aligned}
I_{(0,1)}[L^0 b^1(X_n)]_{t_n,t_{n+1}} &= \left(a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\right)(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdW_t \\
&= \left(a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\right)(X_n)\left(\Delta t_n \Delta W_{t_n} - \Delta Z_{t_n}\right) \\
I_{(0,0)}[L^0 a(X_n)]_{t_n,t_{n+1}} &= L^0 a(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdt \\
&= \left(a\frac{\partial a}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 a}{\partial x^2}\right)(X_n)\frac{1}{2}(\Delta t_n)^2 \\
I_{(1,1,1)}[L^1 L^1 b^1(X_n)]_{t_n,t_{n+1}} &= L^1 L^1 b\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\int_{t_n}^{s} dW_r dW_s dW_t \\
&= \left(b\left(\frac{\partial b}{\partial x}\right)^2 + b^2\frac{\partial^2 b}{\partial x^2}\right)(X_n)\left(\frac{(\Delta W_{t_n})^2}{6} - \frac{\Delta t_n}{2}\right)\Delta W_{t_n}
\end{aligned}
$$

Strong Euler-Maruyama Scheme (1955), $\gamma = 0.5$

Consider the **Geometric Brownian Motion**, which satisfies the SDE

$$\boxed{\begin{aligned} X_0 &= \xi \\ dX_t &= \lambda X_t dt + \mu X_t dW_t \end{aligned}}$$

where λ is called the percentage drift and μ is called the percentage volatility. The exact solution is

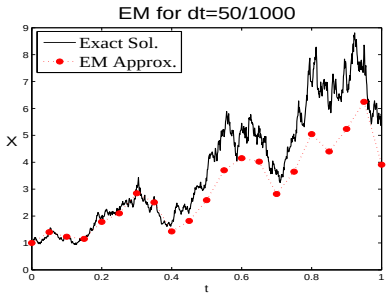
$$X_t = X_0 \exp\left((\lambda - \frac{\mu^2}{2})t - \mu W_t\right)$$

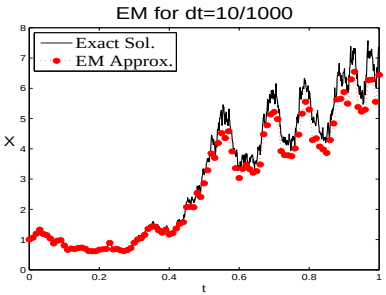
Numerical solution: for $\mathcal{A}_{0.5} = \{v, (0), (1)\}$ (i.e $\ell_1 = 1$), we define the **Euler-Maruyama** scheme

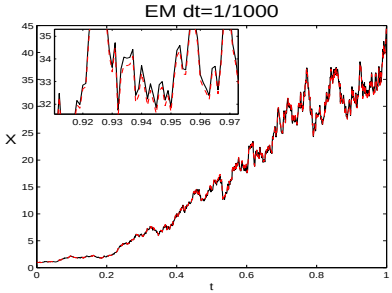
$$\boxed{\begin{aligned} X_0 = Y_0 &= \xi \\ Y_{n+1} &= Y_n + \lambda Y_n \Delta t_n + \mu Y_n \Delta W_{t_n} \end{aligned}}$$

Where Y_n in the numerical approximation of the exact solution X_{t_n}.

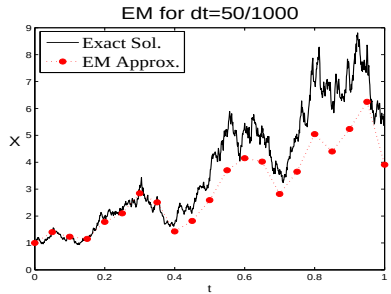
Numerical Simulation of GBM by using EM.

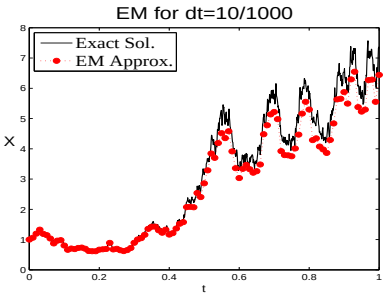


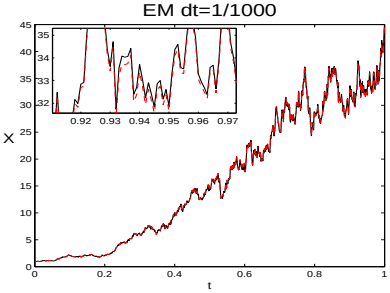


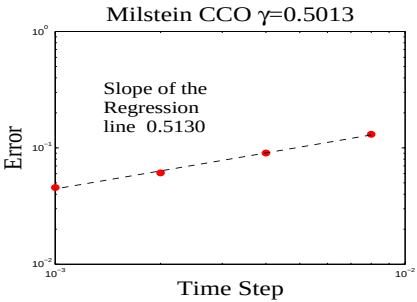


Numerical Simulation of GBM by using EM.









Where $dt = 10^{-3}$ is used to simulate exact solution.

General form of EM scheme

Consider the SDE

$$\boxed{\begin{aligned} X_0 &= \xi \\ dX_t &= a(t, X_t)dt + b(t, X_t)dW_t \end{aligned}}$$

Numerical solution: for $\mathcal{A}_{0.5} = \{v, (0), (1)\}$ (i.e $\ell_1 = 1$), we define the **Euler-Maruyama** scheme

$$\boxed{\begin{aligned} X_0 = Y_0 &= \xi \\ Y_{n+1} &= Y_n + a(t_n, Y_n)\Delta t_n + b(t_n, Y_n)\Delta W_{t_n} \end{aligned}}$$

Where Y_n in the numerical approximation of the exact solution X_{t_n} and

$$\begin{aligned} \Delta t_n &= \int_{t_n}^{t_{n+1}} dt \\ \Delta W_{t_n} &= \int_{t_n}^{t_{n+1}} dW_s. \end{aligned}$$

Stochastic Milstein Scheme (1974), $\gamma = 1$

For $\mathcal{A}_1 = \{v, (0), (1), (1,1)\}$ (i.e $\ell_1 = 2$), we define the **Milstein** scheme

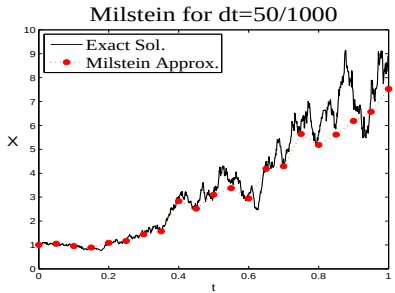
$$\boxed{\begin{aligned} X_0 &= Y_0 = \xi \\ Y_{n+1} &= Y_n + a(t_n, Y_n)\Delta t_n + b(t_n, Y_n)\Delta W_{t_n} \\ &\quad + \frac{1}{2} b(t_n, Y_n)\frac{\partial b}{\partial x}(t_n, Y_n))((\Delta W_{t_n})^2 - \Delta t_n) \end{aligned}} \tag{17}$$

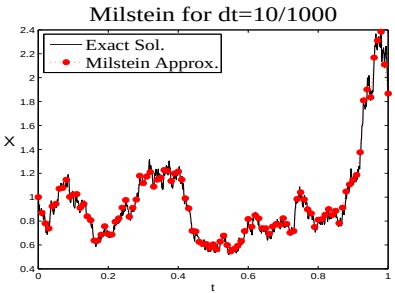
Where Y_n in the numerical approximation of the exact solution X_{t_n} and the additional (to E.M) is

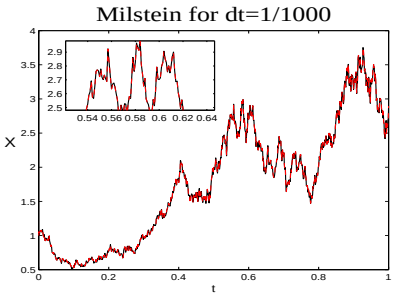
$$f_{(1,1)} I_{(1,1)} \quad = \quad L^1 b(t_n, Y_n) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s^1 dW_t^1.$$

The Milstein method is a first order Itô-Taylor method. It is derived from a truncation of the stochastic Itô-Taylor expansion of the solution.

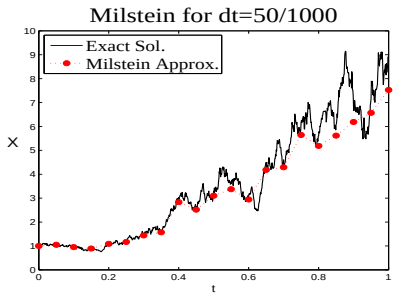
Numerical Simulation of GBM by using Milstein.

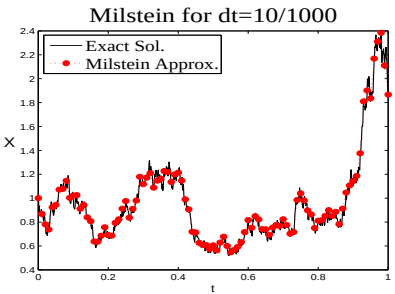


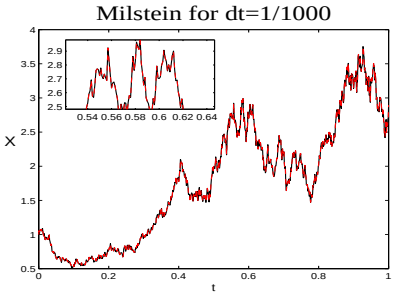


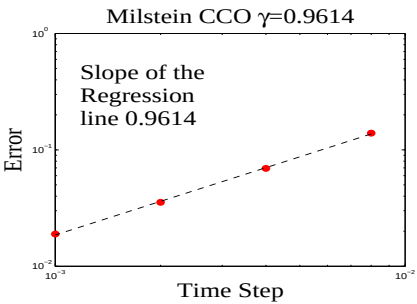


Numerical Simulation of GBM by using Milstein.









Where $dt = 10^{-3}$ is used to simulate exact solution.

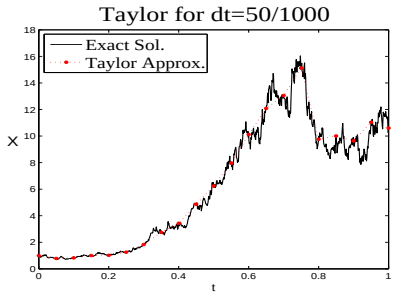
Stochastic Taylor Scheme with order $\gamma = 1.5$

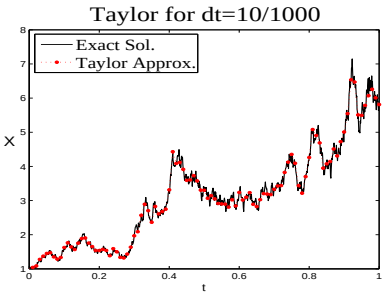
For $\mathcal{A}_{1.5} = \{v, (0), (1), (1,1), (0,0), (1,0), (0,1), (1,1,1)\}$ (i.e $\ell_1 = 3$), we define the **Itô-Taylor** scheme with order $\gamma = 1.5$:

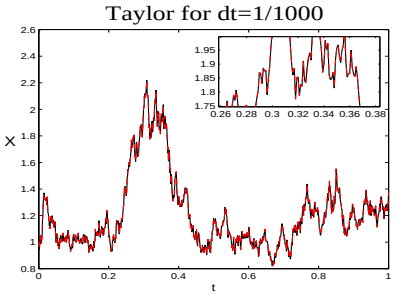
$$
\begin{aligned}
X_0 &= Y_0 = \xi \\
Y_{n+1} &= Y_n + a\Delta t_n + b\Delta W_{t_n} + \frac{1}{2} b \frac{\partial b}{\partial x}((\Delta W_{t_n})^2 - \Delta t_n) \\
&\quad + \frac{1}{2}\left(a\frac{\partial a}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 a}{\partial x^2}\right)(\Delta t)^2 + b\frac{\partial a}{\partial x}\Delta Z_{t_n} \\
&\quad + \left(a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\right)\left(\Delta t_n \Delta W_{t_n} - \Delta Z_{t_n}\right) \\
&\quad + \frac{1}{2}\left(b\left(\frac{\partial b}{\partial x}\right)^2 + b^2\frac{\partial^2 b}{\partial x^2}\right)\left(\frac{1}{3}(\Delta W_{t_n})^2 - \Delta t_n\right)\Delta W_{t_n}
\end{aligned}
\tag{18}
$$

Where Y_n in the numerical approximation of the exact solution X_{t_n} and ΔZ_{t_n} is a normal distributed random variable $\Delta Z_{t_n} \sim N(0, \frac{1}{3}(\Delta t_n)^3)$

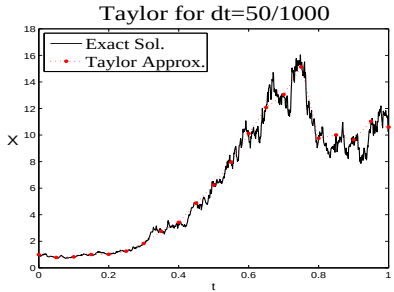
Numerical Simulation of GBM by using Taylor.

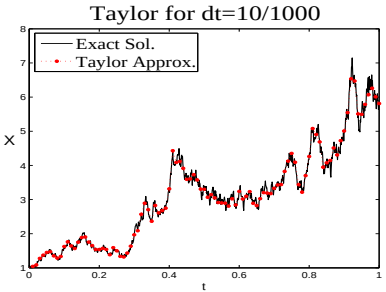


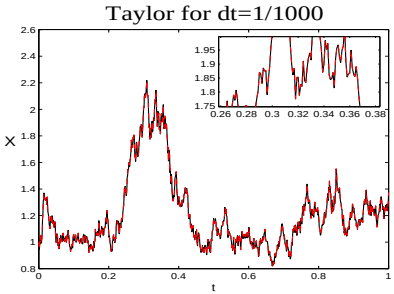


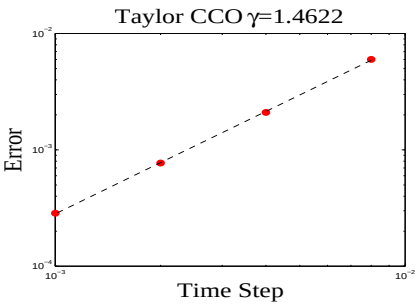


Numerical Simulation of GBM by using Taylor.



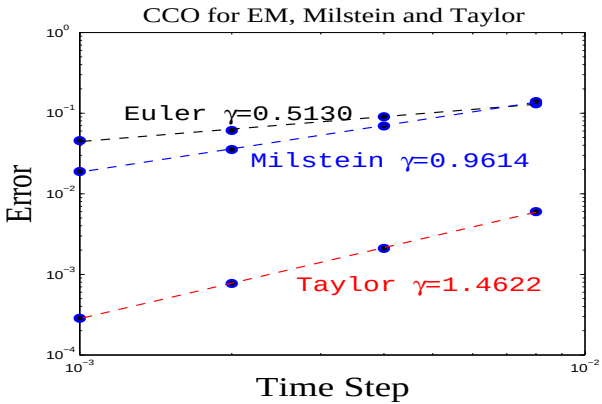






Where $dt = 10^{-3}$ is used to simulate exact solution.

Computational Convergence Orders



Increasing the number of discretization time-points and the number of simulations leads to the best estimation of computational convergence orders. In our cases, we have used a time step $dt = 10^{-3}$ and $M = 1000$ simulation for each numerical simulation.

Concluding Remarks

- The Milstein scheme converges with strong order $\gamma = 1$ and weak order $\beta = 1$. It has a higher order of convergence in the strong sense than the Euler-Maruyama scheme, but gives no improvement in the weak sense.
- The Milstein scheme does **not correspond to a stochastic Taylor scheme** for an hierarchical set for the week convergence.
- **Pathwise convergence** was already considered for RODEs and is also interesting for SODEs because numerical calculations of the random variables Y_n in the numerical scheme above are carried out **path by path**. (P.E. Kloeden and A. Neuenkirch, The pathwise convergence of approximation schemes for stochastic differential equations, LMS J. Comp. Math. 10 (2007))
- Itô stochastic calculus is, however, an L^2 or a **mean-square calculus** and not a pathwise calculus.
- For the week convergence and their construction especially Hierarchical sets, we suggest the monograph of *Kloeden, Peter E., Platen, Eckhard* **Numerical Solution of Stochastic Differential Equations**.

Brownian motion derivatives: White and colored noises

$Itô - Integral$

$$\frac{1}{2}((\Delta W_{t_i})^2 - \Delta t_i) = \int_{t_i}^{t_{i+1}} W_s dW_s$$

Gaussian (White Noise)

$$'\frac{d}{dt}W'_t = \eta_t$$

$$\eta_t \sim \frac{1}{\sqrt{\Delta t}}\mathcal{N}(0,1)$$

W_t

BM increment

$$\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}$$

$$\Delta W_{t_i} \sim \mathcal{N}(0, \sqrt{\Delta t_i})$$

Gaussian (Random Variable)

$$\int_{t_i}^{t_{i+1}} W_s ds = Z_{\Delta_{t_i}}$$

$$Z_t \sim \sqrt{\frac{t^3}{3}}\mathcal{N}(0,1)$$

The white noise as a stationary process has the following properties:

$$E(\xi_{\Delta t}) = 0; \quad Var(\xi_{\Delta t}) = \frac{1}{\Delta t}; \; Cov\Big(\xi_{\Delta t}(t), \xi_{\Delta s}(s)\Big) = 0; \quad \text{if } t \neq s$$

where $\delta_{\Delta t}(t)$) is an approximation of the following $\delta-$function: This noise is called white whenever one talks about uncorrelated (or independent) noise at each pixel. White noise is the noise signal whose power spectrum is flat (the Fourier transform of its covariance). Otherwise the noise is called Colored Noise.

Outlook Research and Graduate-Lectures

- More details about stability, consistency and scheme properties
- Implicit schemes and relationship to stability
- Multistep stochastic schemes for SDEs
- Extrapolation (Richardson-like) stochastic schemes for SDEs
- Variance Reduction: Reduction of the number of sample for getting same results
- ...

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