Numerical Methods for SDEs and SPDEs

Computational Approach

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Part I

Stochastic Differentiation and Integration

- *Random ODEs and Stochastic ODEs*
- *Basics in the theory of Stochastic Processes*
- *Computational Construction of the Brownian motion*
- *Itô formula and Stochastic integrals*
- *Stochastic differential equation*
- *Equivalence SODEs, RODEs and Stratonovich ODEs*

Taylor Expansion: Source of Num. Approximations

- The Taylor expansion of a real function $f(x)$ that is infinitely differentiable at a real x_0 is

$$f(x) = \underbrace{\sum_{j \in A} \frac{(x-x_0)^j}{j!} f^{(j)}(x_0)}_{Approximation\ Term\ P_n(x)} + \underbrace{\sum_{j \in \mathcal{B}(\mathcal{A})} \frac{(x-x_0)^j}{j!} f^{(j)}(x_0)}_{Remainder\ Term} \quad (1)$$

where $\mathcal{A} := \{0, 1, \ldots, p\}$ and $\mathcal{B}(\mathcal{A})$ represents the 'classical' complement of $\mathcal{A}$ in $\mathbb{N}_0$.

- The stochastic Taylor-Expansion of a real valued stochastic process function $f(t, X_t)$ at (τ, X_τ)

$$f(\tau, X_\tau) = \underbrace{\sum_{\alpha \in \mathcal{A}} I_\alpha[f_\alpha(\rho, X_\rho)]}_{Approximation\ Term} + \underbrace{\sum_{\alpha \in \mathcal{B}(\mathcal{A})} I_\alpha[f_\alpha(., X.)]_{\rho,\tau},}_{Remainder\ Term} \quad (2)$$

where ρ and τ are Stop-times satisfying ..., X_t is an adapted Itô process, $\mathcal{A}$ is an index set of the main approximation term and $\mathcal{B}(\mathcal{A})$ represents the complement of $\mathcal{A}$ in a corresponding index set.

Taylor Expansion by using Differential Operators

Starting from the ODE $x'(t) = f(t, x(t))$, we write the deterministic Taylor by using differential operators

$$\begin{aligned} x''(t) &= \frac{d}{dt} f(t, x(t)) = Lf(t, x(t)) \\ x'''(t) &= \frac{d}{dt} Lf(t, x(t)) = L^2 f(t, x(t)) \\ &\vdots \\ x^{(j)}(t) &= L^{j-1} f(t, x(t)), \quad j = 1, 2, \dots \\ L^0 f(t, x(t)) &= f(t, x(t)); \qquad L^{-1} f(t, x(t)) = x(t) \end{aligned}$$

Hence, the Taylor expansion of $x(t)$ on the neighborhood of t_{n+1} for a given discretization $t = t_0 < t_1 < \cdots < t_N = T; h_n = \Delta t_n$ is given by

$$\boxed{x(t_{n+1}) = \sum_{j \in \mathcal{A}} L^{j-1} f(t_n, x(t_n)) \frac{h^j}{j!} + \sum_{j \in \mathcal{B}(\mathcal{A})} L^{j-1} f(t_n, x(t_n)) \frac{h^j}{j!}} \qquad (3)$$

where $\mathcal{A} := \{0, 1, \dots, p\}$ and $\mathcal{B}(\mathcal{A})$ represents the 'classical' complement of $\mathcal{A}$ in $\mathbb{N}_0$.

Random and Stochastic ODEs

- Random ordinary differential equations (RODEs) are pathwise ODEs that contain a stochastic process in their vector field functions.

$$x'(t) = f(\xi(\omega), x(t)) \Longleftrightarrow x'_\omega(t) = g_\omega(t, x(t)) \quad \text{for} \quad \omega \in \Omega \quad (a.s)$$

- Clearly, once ω is fixed, all classical numerical schemes for ODEs are pathwise applicable for RODEs.
- Stochastic ordinary differential equations (SODEs) or (SDEs) are differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is itself a stochastic process.
- Consequently, classical numerical schemes for ODEs are not directly applicable for SDE.

SO, we need stochastic Differentiation and Integration.

The Road to the Computational Stohastics

- Goal: Approximation Methods for Stochastic Processes.
- The stochastic processes, solution of Stochastic ODEs, are continuous BUT! Not differentiable!
- How could, we use the stochastic Taylor expansions of solutions with a new concept of differentiability!
- How could we define the differentiability for stochastic processes!
- The intervention of Itô formula is a good start! Especially in integral form.
- The Itô-Taylor expansions have sense by using some appropriate modifications.

We start by setting up some basics on the Theory of Stochastic Calculus.

Sigma-Algebra

Let Ω be a non-empty set and denote by $\mathcal{P}(\Omega)$ the power set. The set $\mathcal{A} \subseteq \mathfrak{P}(\Omega)$ is called σ-Algebra, if it satisfies the following conditions:

(i) $\Omega \in \mathcal{A}$,

(ii) $B \in \mathcal{A} \Rightarrow B^c \in \mathcal{A}$,

(iii) $B_i \in \mathcal{A},\ i = 1, 2, 3, ... \Rightarrow \bigcup_i B_i \in \mathcal{A}$.

Example

(i) If $\Omega \subset \mathbb{N}$ then $\{\phi, \Omega\}$ and $\mathcal{P}(\Omega)$ are sigma-algebras of subsets of Ω.

(ii) If $\Omega = \{1, 2, 3\}$ then $\mathcal{A} = \Big\{1, \{1, 2\}, \Omega, \{2\}, \{1, 3\}, \phi, \{3\}, \{2, 3\}\Big\}$.

(ii) If $\Omega = \mathbb{R}$ then the set $\mathcal{A}$ of all sets of the form $[a, b], (a, b], (a, b)$ and $[a, b)$ is a sigma-algebra, where $a, b \in \mathbb{R}$.

Probability Map

A probability measure P is a map with the following proprieties:

(i) $P : \mathcal{A} \to [0, 1]$, (**positivity property**.

(ii) $P(\Omega) = 1$, (**sure event**).

(iii) For any pairwise disjoint $(B_i)_{i \in N} \in \mathcal{A}$, it holds $P\Big(\bigcup_{i \in \mathbb{N}} B_i\Big) = \sum_{i \in \mathbb{N}} P(B_i)$. (**countably additive** (a.k.a. **sigma-additive**)).

- The properties above are usually referred to the **Kolmogorov Axioms**.
- A probability map is also called a **probability function**.
- The probability of an **impossible event** is always zero 0 :The probability of getting a number greater than 6, when a die is thrown once, is an **impossible event**, as the highest number in a die is 6.

Probability Space

- The Tuple $(\Omega, \mathcal{A})$ is called **measurable space**
- The Triple $(\Omega, \mathcal{A}, P)$ is called **probability space**.
- The real valued map $X : \Omega \to \mathbb{R}$ is called **random variable**, if X is $\mathcal{A}-$measurable: For all $t \in \mathbb{R}$, it holds

$$\Big\{X \leqslant t\Big\} := X^{-1}\Big(]-\infty, t]\Big) = \Big\{\omega \in \Omega : X(\omega) \in]-\infty, t]\Big\} \in \mathcal{A},$$

- If $d \in \mathbb{N}$ and $X^1, \ldots, X^d$ are random variables defined on the same P.S., the $X = (X^1, \ldots, X^d)$ will be called **random vector**.

Filtration

A family $(\mathcal{F}_t)_{t\in\mathbb{T}}$ ($\mathbb{T}=\mathbb{N}$ or $[0,\infty)$) of σ-Algebras is called **filtration** if the following properties are satisfied:

(i) for all $t\in\mathbb{T}$, $\mathcal{F}_t\subset\mathcal{A}$, (**measurability property**)

(ii) for all $s,t\in\mathbb{T}=\mathbb{N}: s\leqslant t \Rightarrow \mathcal{F}_s\subseteq\mathcal{F}_t$, (**increasing measurable information**)

It is important to know that:

- A filtration $\mathcal{F}$ is right continuous if $\forall\ t\in\mathbb{T},\ \mathcal{F}_t=\bigcap\limits_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$,
- $\mathcal{F}_0$ contains the zero-set A, if $\{A: A\in\mathcal{A}, P(A)=0\}\subseteq\mathcal{F}_0$.

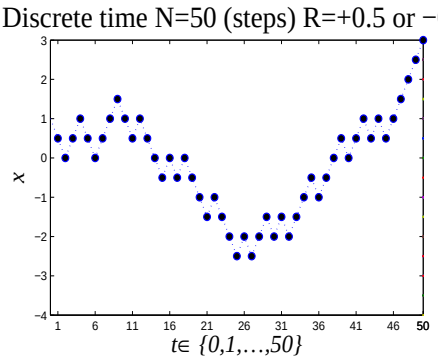
Stochastic Processes

Consider a probability space $(\Omega, \mathcal{A}, P)$ and an index set $\mathbb{T} \subseteq \mathbb{R}^+$.

- The map $X : \mathbb{T} \times \Omega \to \mathbb{R}$ is called **stochastic process** if $X_t : \Omega \to \mathbb{R}$ is a random variable for each $t \in \mathbb{T}$. $\big(X_t(\omega) \equiv X(t, \omega)\big)$.
- If $\mathbb{T} = \mathbb{N}_0$ then X_t is called a **discrete time** stochastic process.
- If $\mathbb{T} = \mathbb{R}+$ then X_t is called a **continuous time** stochastic process.
- A d-**dimensional stochastic process** is a family of d-dimensional random vectors $(X_t)_{t\in\mathbb{T}}$. $(d \in \mathbb{N})$.
- The process $(X_t)_{t\in\mathbb{T}}$ is measurable, if the process $\mathbb{T} \times \Omega \to \mathbb{R}^d$, $(t, \omega) \mapsto X_t(\omega)$ is $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}) - \mathcal{B}(\mathbb{R}^d)$ measurable.
- The process $(X_t)_{t\in\mathbb{T}}$ is called **progressive measurable** with respect to the filtration $\mathcal{F}$, if for any $t \in \mathbb{T}$ the process $[0, t] \times \Omega \to \mathbb{R}^d$, $(s, \omega) \mapsto X_s(\omega)$ is $(\mathfrak{B}([0, t]) \otimes \mathfrak{F}_t) - \mathfrak{B}(\mathbb{R}^d)$ measurable.

Sample of discrete-time Stochastic Processes

For any fixed $\omega \in \Omega$ the function $\mathbb{T} = \mathbb{N}_0 \to \mathbb{R}^d, \quad t \mapsto X_t(\omega)$ is called a **Realization** or **Trajectory** of the process $(X_t)_{t \in \mathbb{T}}$

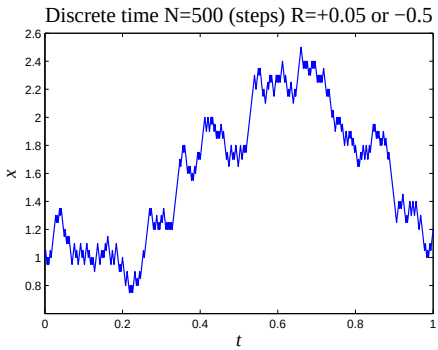


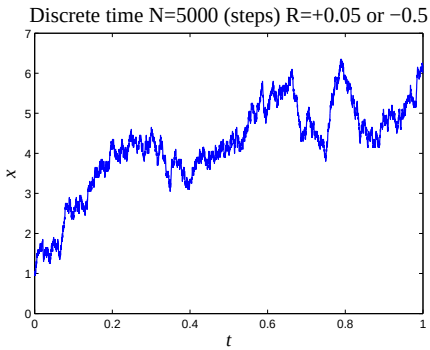
- Start point $X_0 = 1$ (Not random)
- The process X_t is generated by using a Bernoulli Distribution B(0.5).
- $X_{t+1} = X_t + R(\sim B(0.5))$.
- $R = +0.5$ or -0.5 up-down

The figure above shows three different realizations of the time discrete stochastic process X_t.

Sample of continuous-time Stochastic Processes

For any fixed $\omega \in \Omega$ a continuous **Realization** of the process $(X_t)_{t\in[0,1]}$ is constructed by generating random numbers for a discretization of the continuous time-interval $[0, 1]$.





- For the left-figure $\Delta t = 1/500$ and for the right-figure $\Delta t = 1/5000$.
- For $t \in (t_0 + j\Delta t, t_0 + (j + 1)\Delta t)$ linear interpolation is applied.
- Continuous means in the computational sense $\Delta t << 1$.

Martingale and Stop time processes

Let be a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$ and a reel process $X = (X_t)_{t\in\mathbb{T}}$ such that
(i) X is $\mathcal{F}$ adapted (i.e. $\forall t \in \mathbb{T} : X_t$ is $\mathcal{F}_t - measurable$), and
(ii) for all $t \in \mathbb{T} : E(|X_t|) < \infty$.

- X is called **martingale** with respect to the filtration $\mathcal{F}$, if

$$E(X_t|\mathcal{F}_s) = X_s \qquad (a.s)$$

- A map $\rho : \Omega \to \mathbb{T} \cup \{\infty\}$ is called **stop time** with respect to $(\mathcal{F}_t)_{t\in\mathbb{T}}$, if

$$\forall\, t \in \mathbb{T} : \{\rho \leqslant t\} \in \mathcal{F}_t.$$

Brownian motion (standard wiener process)

A one-dimensional **Brownian motion** (also called **standard Wiener process**) is a real-valued stochastic process $\{W_t\}_{t\geqslant 0}$, defined on a probability space $(\Omega, \mathcal{A}, P)$ with filtration $\mathcal{F}$, indexed by nonnegative real numbers t with the following properties:

1. $W_0 = 0$.
2. With probability 1, the function $t \longmapsto W_t$ is continuous in t.
3. The process $\{W_t\}_{t>0}$ has stationary, independent increments.
4. The increment $W_t - W_s$ is normally distributed with mean zero and variance $t - s$ i.e

$$W_t - W_s \sim \sqrt{t-s}\mathcal{N}(0,1), \quad \text{for all} \quad t > s.$$

m-dimensional Brownian motion

For $t \in \mathbb{T}$, $m \in \mathbb{N}$, a m-**dimensional Brownian motion** is given as:

$$W_t = \left(W_t^1, \ldots, W_t^m\right)^T,$$

where

- For all $i = 1, \ldots, m$, $(W_t^i)_{t \in T}$ are independent one dimensional Brownian motions.
- For $0 \leqslant t_0 < \ldots < t_{n-1} < t_n$ the increments $(W_{t_1} - W_{t_0}), \ldots, (W_{t_n} - W_{t_{n-1}})$ are independent identicaly distributed (i.i.d)

The Brownian motion was first discovered by the Scottish botanist Robert Brown in 1827. The notion that the increments of the Brownian motion are normally distributed is the source of immense scientific results, either in stochastic analysis or in the interpretation of physical, biological, econometric models. We refer the reader to [1].

Box Muller transform

The Box-Müller method takes independent standard uniform random variables U_1 and U_2 and produces independent standard normals X_1 and X_2 using the formulas:

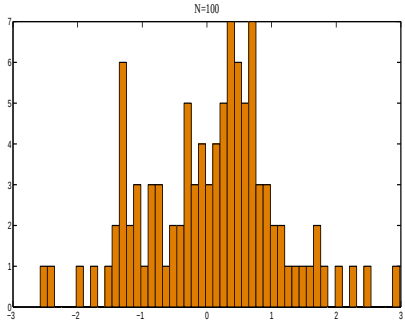
$$\theta = 2\pi U_1, \quad R = \sqrt{-2\ln(U_2)}, \quad X_1 = R\cos(\theta), \quad X_2 = R\sin(\theta). \tag{4}$$

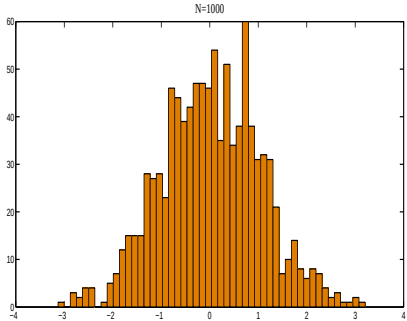
In other words from two random numbers u_1 , $u_2 \in (0, 1]$ (generated by a uniform distribution), we produce two independent standard normally distributed numbers n_1 and n_2, namely:

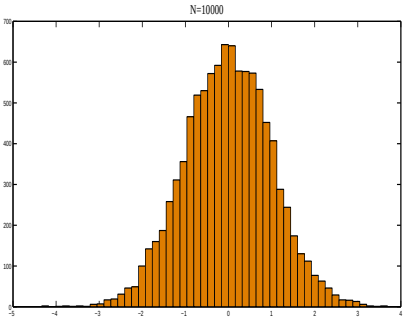
$$n_1 = \sqrt{-2\ln(u_1)}\cos(2\pi u_2), \qquad n_2 = \sqrt{-2\ln(u_1)}\sin(2\pi u_2). \tag{5}$$

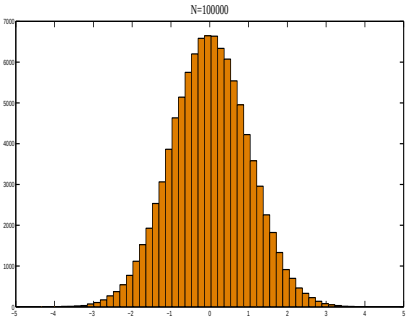
The Box-Müller transform was suggested by George Edward Pelham Box and Mervin Edgar Müller (1958)

Generating random numbers: Box Muller transform









Simulation of the Brownian motion: Algorithm

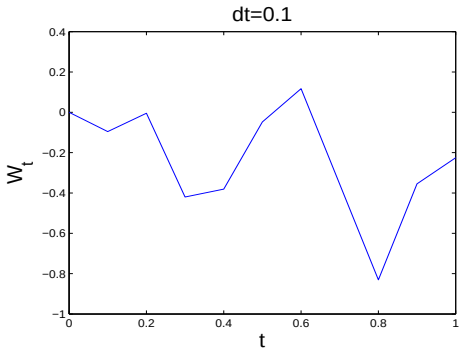
Let $0 = t_0 < t_1 < \ldots < t_N = T$ be an equidistant discretization of the time Interval $[0, T]$, i.e. $t_k = k\Delta$ with $\Delta = \frac{T}{N}$. Since the BM increments are $i.i.d$ and normally distributed. we have

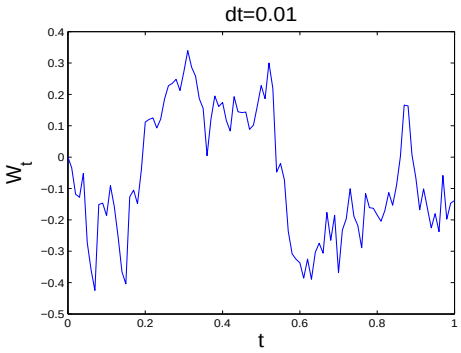
$$\boxed{\begin{aligned} W_{t_0} &= W_0 = 0 \\ W_{t_{k+1}} &= W_{t_k} + R\sqrt{\Delta t} \quad \text{where} \quad R \sim \mathcal{N}(0,1). \end{aligned}}$$

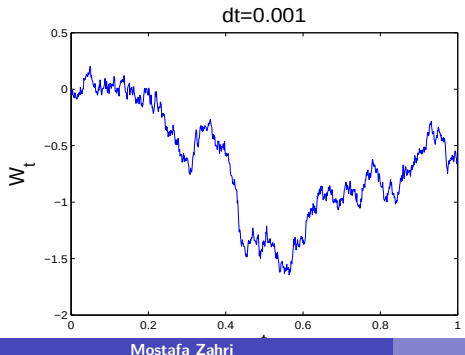
To simulate the path of a Brownian motion

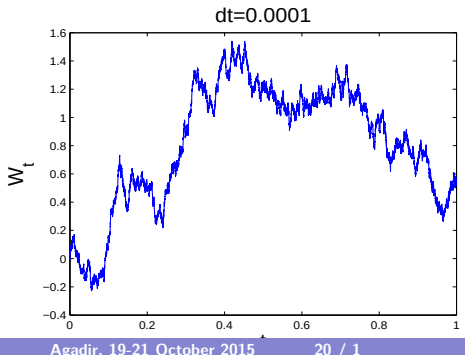
- the values W_{t_k} $\forall k = 0, 1, \ldots, N$ are per recursion obtained,
- for all $t \in]t_k, t_{k+1}[$, W_t are linearly interpolated.

Simulation of the BM.

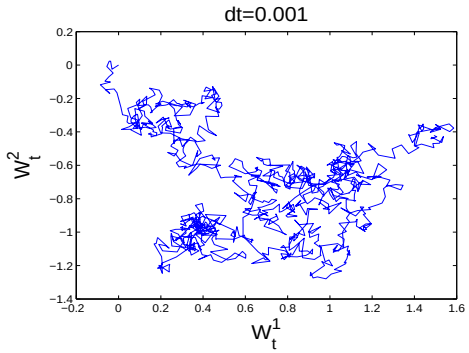


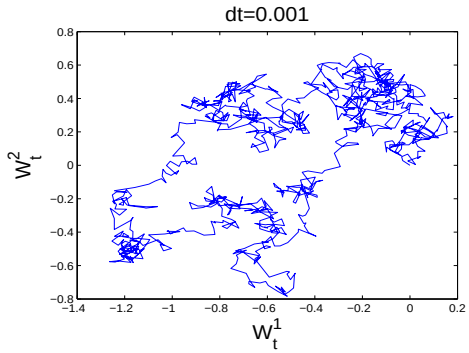


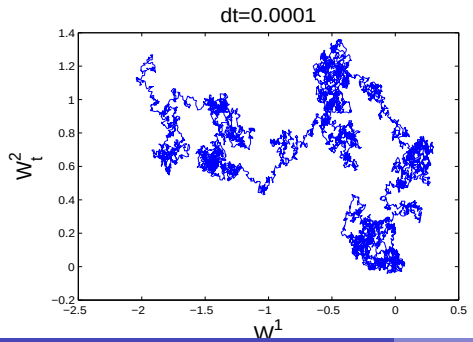


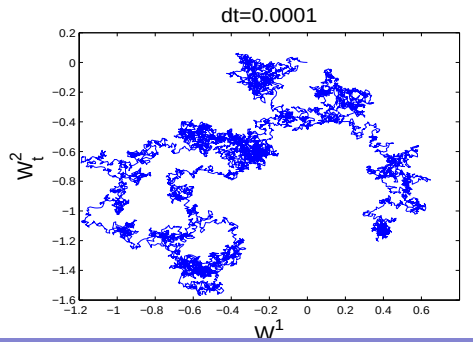


Simulation Planar Brownian motion

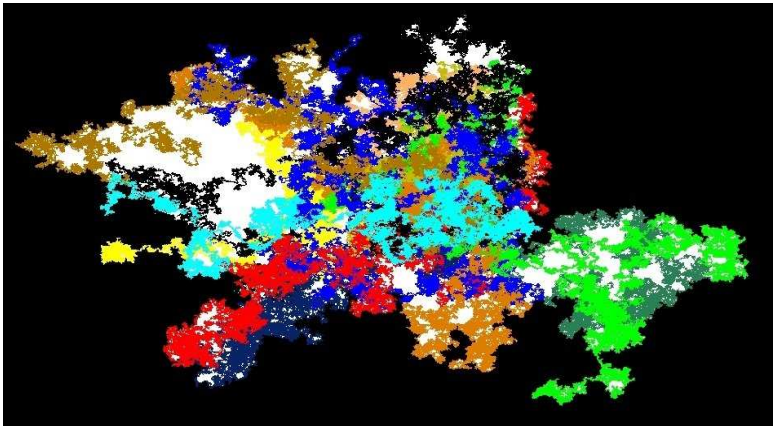








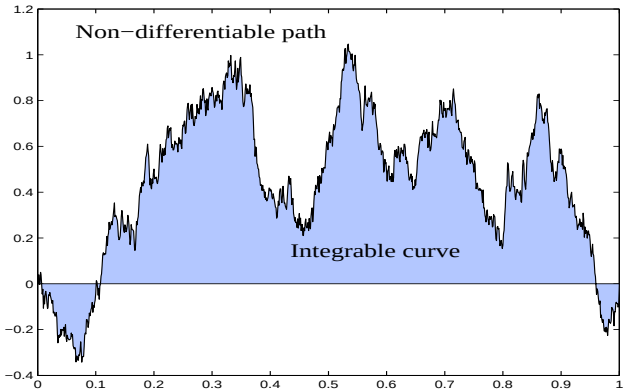
Multi-Planar BM



The two dimensional plot of several planar Brownian motions by using different colors.

Integration instead of differentiation

The integration of the solutions in time is achieved by using the Itô formula in integral form instead of the nondifferentiability (mathematical problem).



The figure above shows a nowhere differentiable stochastic process and a bounded area under the curve.

Stochastic Itô-Integral

Let $(\Omega, \mathcal{A}, P)$ be a probability space with Filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$. For $T > 0$, the set Λ_T is the set of $\mathcal{F}$-adapted stochastic process such that $E\left(\int_0^T Z_s^2 ds\right) < \infty$. The **Itô-Integral** with respect to BM is given as:

$$\boxed{I[Z]_{0,t} := \int_0^t Z_s dW_s = \lim_{N\to\infty} I^{(N)}[Z]_{0,t}}$$

where $Z \in \Lambda_T$ and $I^{(N)}[Z]_{0,t} = \sum_{k=1}^{N} Z_{t_{k-1}^{N,l}} (W_{t_k^{N,l}} - W_{t_{k-1}^{N,l}})$, for $\tau_N^l = \{t_k^{N,l} : k = 0, \ldots, N \text{ und } l \in \mathbb{N}\}$ a discretization sequence of $[0, t]$.

- The limit $\lim_{N\to\infty} E(I[Z]_{0,t} - I^{(N)}[Z]_{0,t})^2 = 0$ exists in $L^2(P)$ and it is unique for all $t \in [0, T]$. (See [7]).

Stochastic Itô-Integral: Useful results

Consider a one dimensional Brownian motion and a non-necessary uniform time discretization $t_k = k\frac{T-t_0}{2^n}$ of the interval $[t_0, T]$. Then we have,

$$1). \quad \lim_{n\to\infty} \sum_{k=0}^{2^n-1} (\Delta t_k)^2 = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \Delta t_k \Delta W_{t_k} = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \Delta W_{t_k} \Delta t_k = 0.$$

$$2). \quad \lim_{n\to\infty} \sum_{k=0}^{2^n-1} (\Delta W_{t_k})^2 = \int_{t_0}^{T} ds = (T - t_0). \qquad \text{(Convergence in } L^2\text{)}$$

where $\Delta t_k = t_{k+1} - t_k$ and $\Delta W_{t_k} = W_{t_{k+1}} - W_{t_k}$.
This means

$$\boxed{dtdt = dtdW_t = dW_t dt = 0},$$

$$\boxed{dW_t dW_t = dt.}$$

Stochastic Itô-Integral: Examples

Using the definition of the Itô integral:

$$\int_0^t dW_s = \lim_n \sum_{i=0}^{n} \Big(W_{t_{i+1}} - W_{t_i}\Big) = W_t - W_0$$

$$\int_0^t W_s dW_s = \lim_n \sum_{i=0}^{n} W_{t_i}\Big(W_{t_{i+1}} - W_{t_i}\Big)$$

$$= \lim_n \sum_{i=0}^{n} W_{t_i} \Delta W_{t_i}$$

$$\boxed{(W_t)^2 - 2W_{t_i}\Delta W_{t_i} = \sum_{i=0}^{n} \Big(\Delta W_{t_i}\Big)^2}$$

$$= \frac{1}{2}(W_t)^2 - \frac{1}{2} \underbrace{\lim_n \sum_{i=0}^{n} \Big(\Delta W_{t_i}\Big)^2}_{\to t \quad (\text{in } L^2)} \qquad \text{later !!!}$$

$$= \frac{1}{2}(W_t)^2 - \frac{1}{2}t$$

Clearly the behavior of the second integral is different from the Riemann-Stieltjes

Itô-Integral proprieties

- **[Linearity]** Consider $(Z_t^{(1)})_{t\in\mathbb{T}}, (Z_t^{(2)})_{t\in\mathbb{T}} \in \Lambda_T$ and $K_1, K_2 \in \mathbb{R}$. For $I[Z^{(1)}]_{0,t} = \int_0^t Z_s^{(1)} dW_s$ and $I[Z^{(2)}]_{0,t} = \int_0^t Z_s^{(2)} dW_s$, it holds

$$\boxed{I[K_1 Z^{(1)} + K_2 Z^{(2)}]_{0,t} = K_1 I[Z^{(1)}]_{0,t} + K_2 I[Z^{(2)}]_{0,t}}$$

- **[Martingale]**

$$\boxed{E(I[Z]_{0,t}|\mathcal{F}_s) = I[Z]_{0,s}, \quad \forall s,t, \quad 0 < s < t.}$$

- **[Itô Isometry]**

$$\boxed{E(I[Z]_{0,t})^2 = E\Big(\int_0^t Z_u^2 du\Big), \quad \forall t, \quad 0 \leqslant t \leqslant T.}$$

- **[Continuity of I_t]** The process $I[Z]_{0,t}$ is continuous with respect to t: It exists a continuous process h_t such that

$$\boxed{P\left(h_t = I[Z]_{0,t}\right) = 1 \quad \forall t, \quad 0 \leqslant t \leqslant T}$$

Multi-indices

For $m \in \mathbb{N}$ and $F = \{0, 1, \ldots, m\}$.For $\alpha = (j_1, \ldots, j_l)$ where $j_i \in F$ and $1 \leqslant i \leqslant l$. We denote the **size** and the **number of zero components**of α

$$\begin{aligned} \ell(\alpha) &:= \ell_\alpha \qquad number\ of\ components\ of\ \alpha \\ n(\alpha) &:= n_\alpha \qquad number\ of\ zero\ components\ of\ \alpha \end{aligned}$$

The set of all multi-indices with respect to F is represented by

$$\mathcal{M} = \bigcup_{l=1}^{\infty} F^l \cup \{v\}, \tag{6}$$

where v refers to the empty multi-index with size zero. The following example gives more sense for the definition above:

Example

For $\alpha = (1, 0, 2)$ it holds $\ell(\alpha) = 3$ and $n(\alpha) = 1$.
For $\alpha = (1, 0, 0, 2, 3, 1, 0, 0)$ it holds $(\alpha) = 8$ and $n(\alpha) = 4$.

Operation on Multi-indices Set

- ($-$ operator). For $\alpha \in \mathcal{M}$ with $\alpha = (j_1, j_2, \ldots, j_l)$. For $l \geqslant 1$, we define $\alpha-$ and $-\alpha$ as follows:

$$\begin{aligned} \alpha- &:= (j_1, j_2, \ldots, j_{l-1}) \quad \text{drop last component.} \\ -\alpha &:= (j_2, \ldots, j_l) \quad \text{drop last component.} \end{aligned}$$

  If $\ell(\alpha) = l > 1$ then, it implies $l(-\alpha) = l(\alpha-) = l - 1$.
  If $\ell(\alpha) = l = 1$ then, it implies $-\alpha = \alpha- = v$ and $l(-\alpha) = l(\alpha-) = 0$.

- ($\star$ operator). Let us consider
  $\alpha = (j_1, j_2, \ldots, j_l), \beta = (i_1, i_2, \ldots, i_k) \in \mathcal{M}$.
  The operator $\star$ is defined as:

$$\begin{aligned} \alpha * \beta &:= (j_1, j_2, \ldots, j_l, i_1, i_2, \ldots, i_k) \quad \text{and} \\ \beta * \alpha &:= (i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_l). \end{aligned}$$

Useful Stochastic Process Sets

Define the set H as a set of stochastic processes $(f_t)_{t\geqslant 0}$, which are progressively adapted to the associated filtration $\{\mathcal{F}_t\}_{t\geqslant 0})$, right continuous and the left limit exists. Moreover, define the sets $H_v, H_{(0)}, H_{(1)}$ as follows

1. $H_v := \{f \in H : \forall\ t \geqslant t_0 \quad |f(t,w)| < \infty \quad \text{a.s.}\}$,
2. $H_{(0)} := \left\{f \in H : \forall\ t \geqslant t_0 \quad \int_{t_0}^{t} |f(s,w)|ds < \infty \quad \text{a.s.}\right\}$,
3. $H_{(1)} := \left\{f \in H : \forall\ t \geqslant t_0 \quad \int_{t_0}^{t} |f(s,w)|^2 ds < \infty \quad \text{a.s.}\right\}$.

For $j \in F\backslash\{0\}$ one sets $H_{(j)} = H_{(1)}$. Here $H_{(\alpha)}$ is defined per recursion as $H_{(\alpha)} := \{f \in H : \quad I_{(\alpha-)}[f(.)]_{t_0,.} \in H_{(j_l)}\}$, for $j_l \in F$ and $l \geqslant 2$.

Useful Notation for multiple Stochastic Integrals

Let us consider $\alpha = (j_1, j_2, \ldots, j_l)$ a multi-index and $(W_t)_{t \geqslant 0}$ an m-dimensional Brownian motion. For $f \in H_{(\alpha)}$, multiple Itô-Integrals are defined per recursion as follows:

$$I_\alpha[f(.)]_{t_0,t} := \begin{cases} f(t), & \text{if } l = 0 \\ \int_{t_0}^{t} I_{\alpha-}[f(.)]_{t_0,s} ds, & \text{if } l \geqslant 1 \text{ and } j_l = 0 \\ \int_{t_0}^{t} I_{\alpha-}[f(.)]_{t_0,s} dW_s^{j_l}, & \text{if } l \geqslant 1 \text{ and } j_l \geqslant 1. \end{cases}$$

Example

$$\begin{aligned} W_t^0 &= t \quad I_{\alpha,t} = I_\alpha[1]_{0,t} \\ I_{(1,2)}[f(.)]_{t_0,t} &= \int_{t_0}^{t} \int_{t_0}^{s} f(z) dW_z^1 dW_s^2, \\ I_{(1,2,0)}[f(.)]_{t_0,t} &= \int_{t_0}^{t} I_{(1,2)}[f(.)]_{0,s} ds = \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{s_1} f(s_2) dW_{s_2}^1 dW_{s_1}^2 ds. \end{aligned}$$

Multiple Stochastic integral

Recurrence relationship for I_α, $\alpha = (j_1, j_2, \ldots, j_n) \in \mathcal{M}$

Theorem

Let us consider $l \in \mathbb{N}$ and $\alpha = (j_1, ..., j_l) \in \mathcal{M}$. For $t \geqslant 0$, we have

$$I_{(j),t} I_{(\alpha),t} = \sum_{i=0}^{l} I_{(\alpha-[l-i])*(j,j_{i+1},...,j_l),t} + \sum_{i=1}^{l} B_{jj_i} I_{(\alpha-[l-i+1])*(0,j_{i+1},...,j_l),t},$$

where $B_{jj_i} = \delta_{j,j_i}(1 - \delta_{0,j})$.

Recurrence relationship for I_α, $\alpha = (j, j, \ldots, j) \in \mathcal{M}$

Corollary

Consider $l, j \in \mathbb{N}$ and $\alpha = (j, j, \ldots, j)$ with $l(\alpha) = l$. It holds:

$$I_{\alpha,t} = \begin{cases} \frac{t^l}{l!} & \text{for } j = 0, \\ \frac{1}{l}(W_t^j I_{\alpha-,t} - t I_{\alpha-[2],t}) & \text{for } j \geqslant 1. \end{cases}$$

Multiple Stochastic integral for $\alpha = (j, j, \ldots, j)$

Lemma

We have the following values of the multiple stochastic integrals for the special case $\alpha = (j, j, \ldots, j) \in \mathcal{M}$:

$$I_{(j,j),t} = \frac{1}{2}(I^2_{(j),t} - t) = \frac{1}{2}\Big((W^j_t)^2 - t\Big). \tag{7}$$

$$I_{(j,j,j),t} = \frac{1}{3}\Big(I_{(j),t}\frac{1}{2}(I^2_{(j),t} - t) - tI_{(j),t}\Big) = \frac{1}{3!}\Big(I^3_{(j),t} - 3tI_{(j),t}\Big), \tag{8}$$

$$I_{(j,j,j,j),t} = \frac{1}{4!}\Big(I^4_{(j),t} - 6tI^2_{(j),t} + 3t^2\Big), \tag{9}$$

$$I_{(j,j,j,j,j),t} = \frac{1}{5!}\Big(I^5_{(j),t} - 10tI^3_{(j),t} + 15t^2I_{(j),t}\Big), \tag{10}$$

$$I_{(j,j,j,j,j,j),t} = \frac{1}{6!}\Big(I^6_{(j),t} - 15tI^4_{(j),t} + 45t^2I^2_{(j),t} - 15t^3\Big), \tag{11}$$

$$I_{(j,j,j,j,j,j,j),t} = \frac{1}{7!}\Big(I^7_{(j),t} - 21tI^5_{(j),t} + 105t^2I^3_{(j),t} - 105t^3I_{(j),t}\Big). \tag{12}$$

Computational value of MSI

If $W_t^{j_1}$ and $W_t^{j_2}$ are two independent Brownian motions the MSI

$$I_{(j_1,j_2)t_0,t} = \int_{t_0}^{t} \int_{t_0}^{s_1} dW_s^{j_1} dW_{s_1}^{j_2},$$

can not be evaluated exactly.
Some proposed approximation could be found in literature

- in [4], one use of the periodic concept of Brownian Bridge and the Fourier series gives another approximation method.
- direct evaluation of the variation given by:

$$\hat{I}_{(j_1,j_2)t_0,t} = \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \int_{t_0}^{t_i} dW_s^{j_1} dW_{s_1}^{j_2} = \sum_{i=0}^{m-1} (W_{t_i}^{j_1} - W_{t_0}^{j_1})(W_{t_{i+1}}^{j_2} - W_{t_i}^{j_2}) \quad (13)$$

- other methods: for instance in Habilitation of Thomas Müller Gronbach [3].

Therefore, $\alpha = (j_1, j_2, \ldots, j_n) \in \mathcal{M}$ for at least $j_i \neq j_k$ components of α the MSI I_α, has no exact analytical expansion, Thus approximation methods are needed.

Itô Process, Stochastic differential equation

Let $(W_t)_{t\in\mathbb{T}}$ be an m-dimensional Brownian motion defined on a $(\Omega,\mathcal{A})^m$ with right continuous augmented filtration $\mathcal{F}=(\mathcal{F}_t)_{t\in\mathbb{T}}$. The process $(X_t^1\ldots,X_t^d)$ is called **Itô Process**, iff it has the following form

$$X_t^i = X_{t_0}^i + \underbrace{\int_{t_0}^t a_s^i ds}_{Riemann-integral} + \underbrace{\sum_{j=1}^m \int_{t_0}^t b_s^{i,j} dW_s^j}_{Itô-integral}, \quad (SDE)$$

where for all $i=1,\ldots,d$, and $j=1,\ldots,m$; the **drift vector** $(a_t^i)_{t\in\mathbb{T}}$ and the **diffusion matrix** $(b_t^{i,j})_{t\in\mathbb{T}}$ are $\mathcal{F}_t$ adapted and satisfy $\int_{t_0}^T a_s^i ds<\infty$ and $\int_{t_0}^T (b_s^{i,j})^2 ds<\infty$ a.s.

- The (SDE) refers to the **stochastic differential equation** in integral form and in differential form is given by

$$dX_t = a(t,X_t)dt + b(t,X_t)dW_t$$

Higher order Itô formula

Let us consider the functional $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}$ with continuous partial derivatives $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x^i}$ and $\frac{\partial^2 f}{\partial x^i \partial x^j}$ for $i = 1, \ldots, d$. Moreover, consider a d-dimensional Itô-Process $(X_t)_{t \in \mathbb{T}}$, then we have,

$$\begin{aligned} f(t, X_t^1, \ldots, X_t^d) = f(t_0, X_{t_0}^1, \ldots, X_{t_0}^d) &+ \int_{t_0}^t \frac{\partial f}{\partial s}(s, X_s^1, \ldots, X_s^d) ds \\ &+ \sum_{i=1}^d \int_{t_0}^t \frac{\partial f}{\partial x^i}(s, X_s^1, \ldots, X_s^d) dX_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^d \int_{t_0}^t \frac{\partial^2 f}{\partial x^i x^j}(s, X_s^1, \ldots, X_s^d) d < X^i, X^j >_s, \end{aligned}$$

where $d < X^i, X^j >_s = \sum_{k=1}^m b^{i,k}(s, X_s) b^{j,k}(s, X_s) ds$

One dimensional Itô formula

The one dimensional case $d = m = 1$ of the Itô-Formula is given as

$$f(t, X_t) = f(t_0, X_{t_0}) + \int_{t_0}^{t} \left\{ \frac{\partial f}{\partial s}(s, X_s) + a_s \frac{\partial f}{\partial x}(s, X_s) + \frac{1}{2} b_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s) \right\} ds + \int_{t_0}^{t} b_s \frac{\partial f}{\partial x}(s, X_s) dW_s.$$

Proof idea: By using the Taylor expansion of order two: (all convergence in L^2)

$$\begin{aligned} f(t, X_t) &= f(t_0, X_{t_0}) + \sum_{k=0}^{2^n-1} \Delta f(t_k, X_{t_k}) \\ &= f(t_0, X_{t_0}) + \underbrace{\sum_{k=0}^{2^n-1} \frac{\partial f}{\partial t} \Delta t_k}_{\to \int_{t_0}^{t} \frac{\partial f}{\partial t} ds} + \underbrace{\sum_{k=0}^{2^n-1} \frac{\partial f}{\partial x} \Delta X_{t_k}}_{\to \int_{t_0}^{t} \frac{\partial f}{\partial x} a_s ds + \int_{t_0}^{t} \frac{\partial f}{\partial x} b_s dW_s} + \frac{1}{2} \underbrace{\sum_{k=0}^{2^n-1} \frac{\partial^2 f}{\partial x^2} (\Delta X_{t_k})^2}_{\to \int_{t_0}^{t} \frac{\partial^2 f}{\partial x^2} b_s^2 ds} \\ &\quad + \underbrace{\sum_{k=0}^{2^n-1} \frac{\partial^2 f}{\partial t \partial x} \Delta t_k \Delta X_{t_k}}_{\to 0} + \underbrace{\frac{1}{2} \sum_{k=0}^{2^n-1} \frac{\partial^2 f}{\partial t^2} (\Delta t_k)^2}_{\to 0} + \underbrace{\sum_{k=0}^{2^n-1} R_k}_{\to 0}. \end{aligned}$$

One dimensional Itô formula: Example

For $f(t,x) = \frac{1}{2}x^2$ with $X_t = W_t$ and $a_t = 0, b_t = 1$. Applying Itô formula:

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + a_t\frac{\partial f}{\partial x}dt + b_t\frac{\partial f}{\partial x}dW + \frac{1}{2}b_t^2\frac{\partial^2 f}{\partial x^2}dt \\ &= \frac{\partial f}{\partial t}dt + (1)\frac{\partial f}{\partial x}dW + \frac{1}{2}(1)^2\frac{\partial^2 f}{\partial x^2}dt \\ &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dW + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt \\ &= \frac{\partial f}{\partial x}dW + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt \end{aligned}$$

Hence $\frac{1}{2}dW_t^2 = W_t dW_t + \frac{1}{2}dt$ and $\frac{1}{2}\int dW_s^2 = \int W_s dW_s + \frac{1}{2}\int dt$ Thus,

$$\boxed{I_t = \int_0^t W_s dW_s = \frac{1}{2}(W_t^2 - t)}$$

We got the same result as by using the variational calculus.

Exact Solution of SDE

Let a (drift) and b (diffusion) are two measurable vector functions defined as follows: $a : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $b : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$. If the $d-$dim SDE with a F_{t_0}-measurable initial condition $X_{t_0} = \Upsilon$ (RV), independent from $\sigma(W)$, satisfies:

1. (Lipschitz condition) For $t \in [t_0, T],\ x, y \in \mathbb{R}^d$ and a constant $K > 0$:

$$\|a(t,x) - a(t,y)\| \leqslant K\|x - y\| \quad \text{and} \quad \|b(t,x) - b(t,y)\| \leqslant K\|x - y\|$$

2. (Linear Growth) For $t \in [t_0, T],\ x \in \mathbb{R}^d$ and a constant $D > 0$:

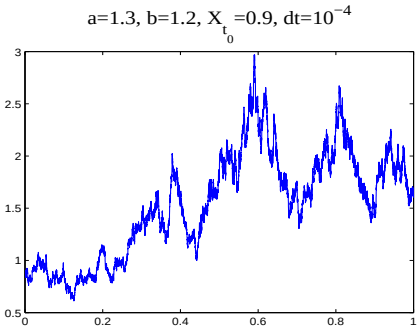
$$\|a(t,x)\|^2 + \|b(t,x)\|^2 \leqslant D(1 + \|x\|^2).$$

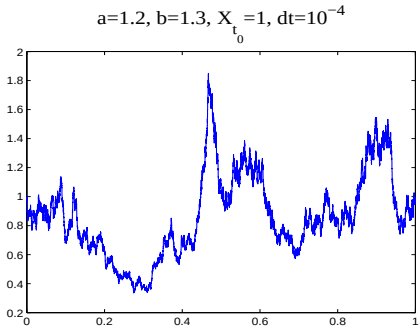
then the SDE has a **pathwise unique solution** X_t on $[t0, T]$ with $E\left(\sup_{t_0 \leqslant s \leqslant T} |X_s|^2\right) < \infty$.

Simulation of Exact Solution of a SDE

Consider the linear SDE: $X_t = aX_t ds + bX_t dW_t$ with exact solution

$$X_t = X_{t_0} \exp\left((a - \frac{1}{2}b^2)t + bW_t\right)$$





Brownian motion derivatives: White and colored noises

$Itô - Integral$

$$\frac{1}{2}((\Delta W_{t_i})^2 - \Delta t_i) = \int_{t_i}^{t_{i+1}} W_s dW_s$$

Gaussian (White Noise)

$$\frac{d}{dt}W_t = \xi_t$$

$$\xi_t \sim \frac{1}{\sqrt{\Delta t}}\mathcal{N}(0,1)$$

W_t

BM increment

$$\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}$$

$$\Delta W_{t_i} \sim \mathcal{N}(0, \sqrt{\Delta t_i})$$

Gaussian (Random Variable)

$$\int_{t_i}^{t_{i+1}} W_s ds = Z_{\Delta_{t_i}}$$

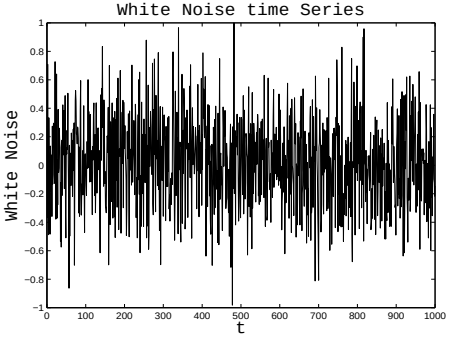
$$Z_t \sim \sqrt{\frac{t^3}{3}}\mathcal{N}(0,1)$$

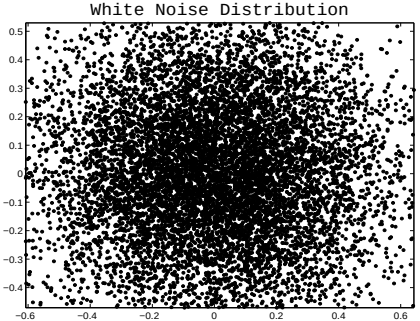
The white noise as a stationary process has the following properties:

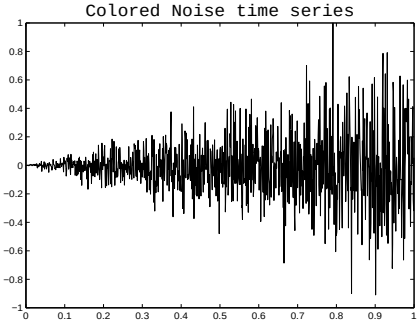
$$E(\xi_{\Delta t}) = 0; \quad Var(\xi_{\Delta t}) = \frac{1}{\Delta t}; \; Cov\Big(\xi_{\Delta t}(t), \xi_{\Delta s}(s)\Big) = 0; \quad \text{if } t \neq s$$

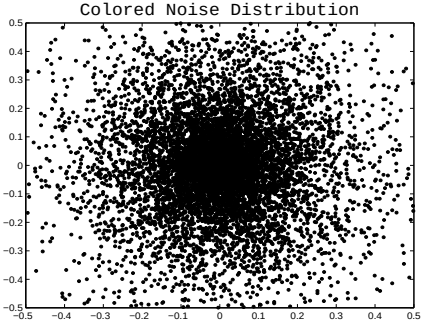
where $\delta_{\Delta t}(t)$) is an approximation of the following $\delta-$function: This noise is called white whenever one talks about uncorrelated (or independent) noise at each pixel. White noise is the noise signal whose power spectrum is flat (the Fourier transform of its covariance). Otherwise the noise is called Colored Noise.

Time series of White and colored Noises









Random and Stochastic ODEs, 'again'

Consider the RODE

$$\boxed{\frac{dx}{dt} = a(x) + b(x)\zeta_t^{\Delta}} \tag{14}$$

- The noise in physical systems usual has a wide band spectrum, i.e. a Δ-correlated stationary Gaussian process ζ_t^{Δ}
- white noise limit as $\Delta \to 0$.
- The RODE (14) leads to the so called Stratonovich[1] SDE

$$\boxed{dX_t = a(X_t) + b(X_t) \circ dW_t} \tag{15}$$

- The Stratonovich SDE with a Wiener process can be rewritten as stochastic differential system of equations

Example

$$\frac{dx}{dt} = -x + \sin(W_t) \Leftrightarrow d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} -X_t + \sin(Y_t) \\ 0 \end{pmatrix} dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dW_t$$

[1] *Wong & Zakai (1965), Godin & Molchanov (2007)*

Equivalence SDEs vs RODEs

In general a stochastic differential equations SODEs can be rewritten as RODEs

$$dX_t = f(X_t)dt + dW_t \Leftrightarrow \frac{dz}{dt} = f(z + O_t) + O_t$$

where O_t is the stochastic stationary Ornstein-Uhlenbeck process satisfying the linear SDE

$$dO_t = -O_t dt + dW_t \qquad \text{and} \qquad z(t) = X_t - O_t$$

The continuity and the fundamental theorem of calculus implies the pathwise differentiability[2] of

$$\begin{aligned} z(t) &= X_t - O_t \\ &= X_0 - O_0 + \int_0^t \Big(f(X_s) + O_s \Big) ds \\ &= z(0) + \int_0^t \Big(f(z(s) + O_s) + O_s \Big) ds \end{aligned}$$

So, We can use deterministic calculus pathwise for RODEs[3]

[2] *Doss, Sussmann (1970s), Imkeller, Lederer, Schmalfuss (2000s)*

[3] *Peter Kloeden and Arnuf Jenzen.*

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