Mathematical modeling of patterns in Islamic design

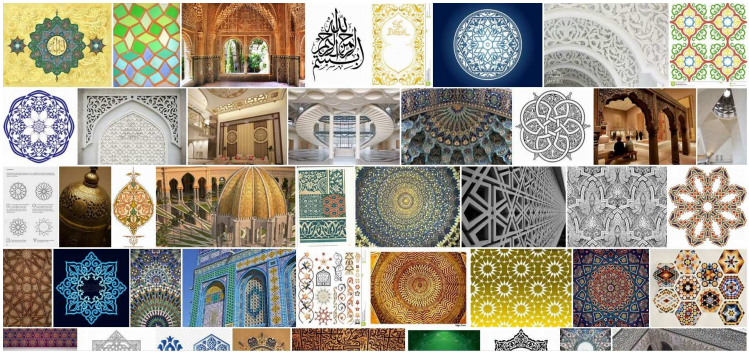
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Islamic Design

Islamic Design in the Web. (search words: islamic design) Related words: *Islamic complex geometry, - geometric pattern, - Design House, - Arts ...*



Zellige is mosaic tilework made from individually chiseled geometric tiles set into a plaster base. This form of Islamic art is one of the main characteristics of Moroccan architecture. It consists of geometrically patterned mosaics, used to ornament walls, ceilings, fountains, floors, pools and tables.
(source: *https://en.wikipedia.org/wiki/Zellige*)

Contents

We mainly discuss the following items:

- Periodic sequences on one-dim manifoldS^1, Definitions and Properties
- Construction of Symmetrical Islamic Design
- Numerical Simulation of Islamic Design

For $x_a(0) = \alpha$, consider the real iterative sequence $x_a : \mathbb{N} \longrightarrow S^1$ mod. 2π by:

$$x_a(0) = \alpha$$
$$x_a(n) := \begin{cases} x_a(n-1) + a\pi & \text{if } n \text{ odd,} \\ x_a(n-1) - (1-a)\pi & \text{if } n \text{ even.} \end{cases}$$

Example of the up-down behavior of the sequence $x_a(n)$ (mod 2π) will be observed as moves of emerging numbers on S^1:

a	$x_a(0)$	$x_a(1)$	$x_a(2)$	$x_a(3)$	$x_a(4)$	$x_a(5)$	$x_a(6)$	$x_a(7)$	$x_a(8)$	$x_a(9)$
0	0	0	$-\pi$	$-\pi$	-2π	-2π	-3π	-3π	-4π	-4π
$\frac{1}{4}$	0	$\frac{1}{4}\pi$	$\frac{-2}{4}\pi$	$\frac{-1}{4}\pi$	$\frac{-4}{4}\pi$	$\frac{-3}{4}\pi$	$\frac{-6}{4}\pi$	$\frac{-5}{4}\pi$	$\frac{-8}{4}\pi$	$\frac{-7}{4}\pi$
$\frac{1}{2}$	0	$\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$
$\frac{3}{4}$	0	$\frac{3}{4}\pi$	$\frac{2}{4}\pi$	$\frac{5}{4}\pi$	$\frac{4}{4}\pi$	$\frac{7}{4}\pi$	$\frac{6}{4}\pi$	$\frac{9}{4}\pi$	$\frac{8}{4}\pi$	$\frac{11}{4}\pi$
1	0	π	π	2π	2π	3π	3π	4π	4π	5π

Observe the first $n \neq 0$, such that $x_a(n) = \alpha$.

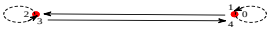
Remarks and Properties of the sequence x_a

According to the example, we remarked:

- The sequence $x_a(n)$ is periodic for special values of the parameter a only.
- For $a = 0.00$, we needed one rotation in the counterclockwise direction and four iteration to reach the first-come-back $X_0(4) = 0$ for $n = 4$ and $k = -1$
- For $a = 0.25$, we needed one rotation in the counterclockwise direction and eight iteration to reach the first-come-back $X_{0.25}(8) = 0$ for $n = 8$ and $k = -1$
- For $a = 0.50$, we consider this case as a critical one. There is a come-back to the origin $x_{0.5}(0) = 0$ without rotating a complete round. It's in fact an oscillatory behavior.
- ...
- For $a = 0.95$ we needed one rotation in the counterclockwise direction and four iteration to reach the first-come-back $X_{0.95}(40) = 0$ for $n = 40$ and $k = 9$

We are interested to characterize the minimal number of rotation $k \in \mathbb{Z}$ and the minimal n till the first-come-back $X_a(n) = x_a(0)$.

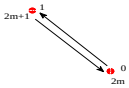
we are looking for relationship between a and the minimal index such that $x_a(n) = \alpha$.



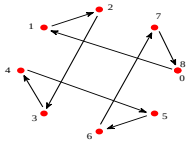
(A) $a = 0;\ k = -1;\ n = 4$



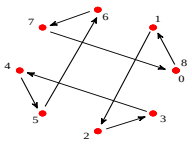
(B) $a = 1;\ k = 1;\ n = 3$



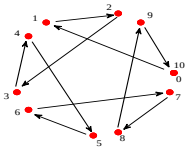
(C) $a = 0.5;\ k = 0;\ n = 2$



(D) $a = 0.75;\ k = 1;\ n = 8$



(E)
$a = 0.25;\ k = -1;\ n = 8$



(F) $a = 0.7;\ k = 1;\ n = 10$

Convergence Theorem and Properties

Theorem

For any rational number $a \in [0,1] \cap \mathbb{Q}$ and any initial value $x_a(0)$, there exist n such that $x_a(n) \equiv x_a(0) \mod 2\pi$ and if $a \notin [0,1] \cap \mathbb{Q}$, we have $x_a(n) \neq x_a(0)$ for all $n \in \mathbb{N}$.

- For the proof. We characterise a as a function of the number of loops k and the first-come back n_f.
- Moreover, we distinguish between the odd and the even come-back events:

Definition

According to theorem 1., we define the come-back mappings mapping (for construction of the parameter a)

$$e(k,n) : \mathbb{Z} \times 2\mathbb{N} \longrightarrow [0,1]; \qquad e(k,n) := \frac{1}{2}\left(\frac{4k}{n} + 1\right). \tag{1}$$

and

$$o(k,n) : \mathbb{Z} \times (2\mathbb{N}+1) \longrightarrow [0,1]; \qquad o(k,n) := \frac{1}{2}\left(\frac{4k-1}{n} + 1\right). \tag{2}$$

Periodicity of x_a

Lemma

Let $k \in \mathbb{Z}$ and n be an even natural number, then the following identities holds

1. $e(-k,n) + e(k,n) = 1$.
2. $o(-k,n) + o(k,n) = 1 - \dfrac{1}{2n}$.

- The first-come-back n is even, making one k loops in the positive direction or in the negative one have the same behavior. Which is not the case if n is odd
- But we could admit the result in the even case for large n only.

Lemma

Let us consider $k \in \mathbb{Z}$ and n in $\mathbb{N}$, then the following identities holds

1. $e(k,n) = e(pk,pn)$ *for all* $p \in \mathbb{Z}$.
2. $o(k,n) = o\left(pk - \dfrac{p-1}{4}, pn\right)$ *for all odd* $p \in \mathbb{N}$ *such that* $(p-1)/4 \in \mathbb{N}$.

The range of $x_{o(k,n)}$ and $x_{e(pk,pn)}$ are identical for all $p \in \mathbb{N}$. But $x_{o(k,n)}$ and $x_{o(pk,pn)}$ are identical only for odd numbers $p \in \mathbb{N}$ such that $(p-1)/4 \in \mathbb{N}$.

Backward given a finding the firs-come back n_f. First recall:

DEFINITION

We define the following indices:

(I) The first-come-bach index n_f is defined as $n_f = \min\{n | x_a(n) = x_a(0)\}$.

(II) If $\text{Range}(x_a) := \{x_a(0), \ldots, x_a(m)\}$, then denote by $n_r = m$ the maximal index in the range of x_a.

(III) The cardinal of the range $Range(x_a)$ of the sequence x_a is defined by $n_c := \sharp$ Range (x_a). (i.e number of distinct elements generated by the sequence).

We have the following theorem:

THEOREM

If $j_e = \min\left\{j |\ j(2a-1)/2 \in \boldsymbol{N}\right\}$ *and* $j_o = \min\left\{j |\ (j(2a-1)+a)/2 \in \boldsymbol{N}\right\}$, *then the first come back* n_f *of* x_a *is given by*

$$n_f = \min\left\{2j_e, 2j_o + 1\right\}. \tag{3}$$

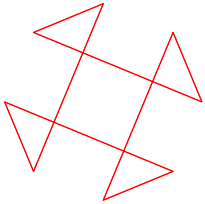
Examples: For $a = 4/5$, the 15 first elements

n	$x_{\frac{4}{5}}(n)$	Remarks
0	0	Initial Element in the range : 1
1	$\frac{4}{5}\pi$	New Element : 2
2	$\frac{3}{5}\pi$	New Element : 3
3	$\frac{7}{5}\pi$	New Element : 4
4	$\frac{6}{5}\pi$	New Element : 5
5	$\frac{10}{5}\pi = 2\pi \equiv 0$	First-come-back and Existent Element $n_f = 5$
6	$\frac{9}{5}\pi$	New Element : 6
7	$\frac{3}{5}\pi$	Existent Element $x_a(7) = x_a(2)$
8	$\frac{2}{5}\pi$	New Element : 7
9	$\frac{6}{5}\pi$	Existent Element$x_a(9) = x_a(4)$
10	π	New Element : 8
11	$\frac{9}{5}\pi$	Existent Element $x_a(11) = x_a(6)$
12	$\frac{8}{5}\pi$	New Element : 9
13	$\frac{2}{5}\pi$	Existent Element $x_a(13) = x_a(8)$
14	$\frac{1}{5}\pi$	New Element : $n_c = 10$ but $n_r = 14$

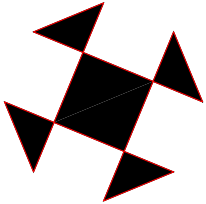
Similarly, we compute the same indices for $a = 3/4$, $a = 2/3$ and for $a = 5/7$:

$index/a$	0	1	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{7}{10}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{5}{7}$
n_f	4	3	∞	8	8	10	5	8	9	7
n_c	2	2	2	8	8	10	10	8	6	14
n_r	4	2	2	8	8	10	14	8	9	20

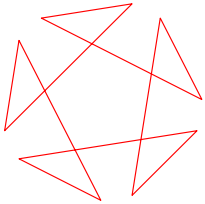
- According to the trajectory (in $\mathbb{R}^2$) of the sequence x_a,
- From closed graphs to colored cells:



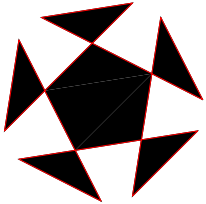
(G) $k = 1;\ n = 8$



(H) $k = 1;\ n = 8$



(I) $k = 1;\ n = 10$



(J) $k = 1;\ n = 10$

Construction of the Islamic Design: The cell

Islamic design employs polygons with vertices on a circle. These polygons represents connected graph $G(a,\alpha)$, where the vertices $V(a,\alpha)$ and edges $E(a,\alpha)$, where α represents the initial value of the sequence x_a^α. For a given parameter a and an initial value α, the vertices set is defined as spatial points in $\mathbb{R}^2$ by:

$$V(a,\alpha) := \left\{p_a^\alpha(0), p_a^\alpha(1), \ldots, p_a^\alpha(n_r)\right\}, \tag{4}$$

where the planar points $p_a^\alpha(t) = p(x_a^\alpha, t) = \left(\cos(x_a^\alpha(t)), \sin(x_a^\alpha(t))\right) \in \mathbb{R}^2$ for $t = 0, 1, \ldots, n_r$.
The set of edges $E(a,\alpha)$ of the closed graph $G(a,\alpha)$ is

$$E(a,\alpha) := \left\{\left(p_a^\alpha(0), p_a^\alpha(1)\right), \left(p_a^\alpha(1), p_a^\alpha(2)\right), \ldots, \left(p_a^\alpha(n_r-1), p_a^\alpha(n_r)\right)\right\}. \tag{5}$$

where the natural number n_r as defined above, $a \in [0,1] \cap \mathbb{Q}$.

Definition (single cell)

A cell $C(q,r,a,\alpha)$ defining design is a graph $G(a,\alpha)$ with colored subareas generated by a sequence $x_a\alpha$ on a circle with radius $r > 0$ and a center $q = (q_x, q_y) \in \mathbb{R}^2$. We denote by $D(\mathbb{R}^2)$, the set of all possible Islamic designs in $\mathbb{R}^2$.

For a given $v, h \in \mathbb{R} \setminus \{0\}$, we define:
The vertical translation mapping f_v as

$$\begin{aligned} f_v \quad : \quad & D(\mathbb{R}^2) \longrightarrow D(\mathbb{R}^2) \\ & f_{(0,v)}(C((q_x, q_y), r, a, \alpha)) = C((q_x, q_y + v), r, a, \alpha). \end{aligned} \tag{6}$$

The horizontal translation mapping f_h is defined by

$$\begin{aligned} f_h \quad : \quad & D(\mathbb{R}^2) \longrightarrow D(\mathbb{R}^2) \\ & f_{(h,0)}(C((q_x, q_y), r, a, \alpha)) = C((q_x + h, q_y), r, a, \alpha). \end{aligned} \tag{7}$$

The diagonal translation mapping $f_{(v,h)}$ is defined by

$$\begin{aligned} f_{(v,h)} \quad : \quad & D(\mathbb{R}^2) \longrightarrow D(\mathbb{R}^2) \\ & f_{(v,h)}(C((q_x, q_y), r, a, \alpha)) = C((q_x + h, q_y + v), r, a, \alpha), \end{aligned} \tag{8}$$

where $h \neq 0$ and $v \neq 0$.

The Design (space filling) mapping $F_{(m_v,m_h)}$ is defined by

$$\begin{aligned} F_{(v,h)} \quad &: \quad D(\mathbb{R}^2) \longrightarrow D(\mathbb{R}^2) \\ & F_{(m_v,m_h)}(C(q,r,a,\alpha)) = \sum_{h=1}^{m_h}\sum_{v=1}^{m_v} f_{(K_v v, K_h h)}(C((q_x,q_y),r,a,\alpha)), \end{aligned} \tag{9}$$

where K_v, K_h are positive constants.
The radial translation mapping f_s is defined by

$$\begin{aligned} f_s \quad &: \quad D(\mathbb{R}^2) \longrightarrow D(\mathbb{R}^2) \\ & f_s(C(q,r,a,\alpha)) = C(q,sr,a,\alpha). \end{aligned} \tag{10}$$

We suggest the following mathematical definition of islamic designs:

DEFINITION (ISLAMIC DESIGN)

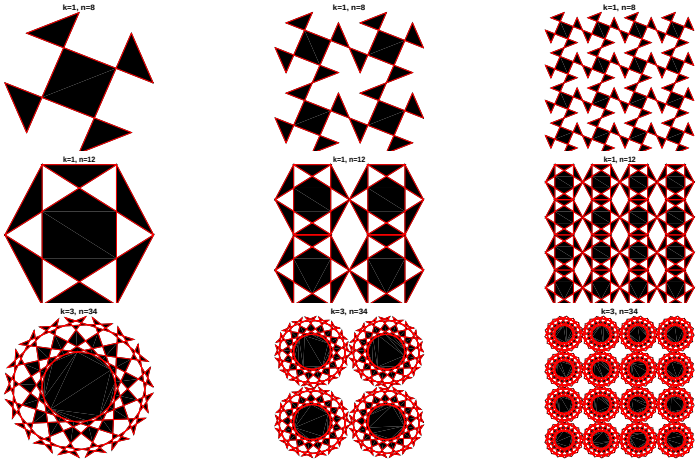
An Islamic Design $D(m_v, m_h, C(q,r,a,\alpha))$ with $m_v \times m_h \in \mathbb{N}^2$ elements is a symmetrical repeated placement of cells of the form $C(q,r,a,\alpha)$.

We restrict ourself for constructing four symmetrical Islamic Designs:

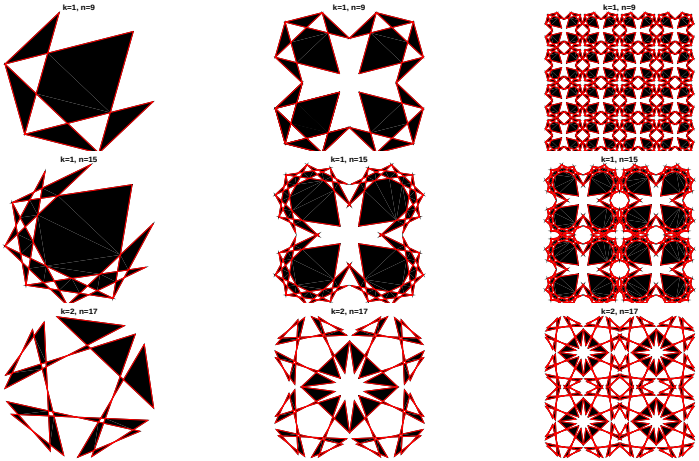
1. Modeling of Islamic design using **single cells**.
2. Modeling of Islamic design using **composite cells**.
3. Modeling of Islamic design using **circular cells**.
4. Modeling of Islamic design using **overlapped cells**.

Just a remark: We can construct more type of Islamic Design by introducing more planar transformations. (Future works).

$$D(m_v, m_h, C(q, r, a, \alpha)) = \sum_{h=1}^{m_h} \sum_{v=1}^{m_v} f_{(Kv,Kh)} C(q, r, a, \alpha).$$

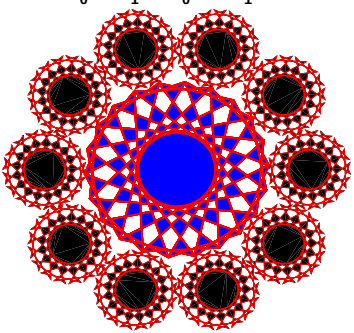


$$D\big(m_v, m_h, D(1,1,C(q^z,r,a,(\alpha_z)))\big) \quad = \quad \sum_{h=1}^{m_h}\sum_{v=1}^{m_v} f_{(Kv,Kh)} D(1,1,C(q^z,r,a,(\alpha_z))),$$

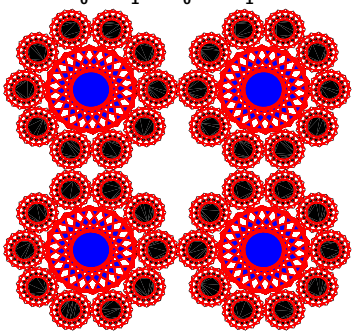


$$D(m_v, m_h, C(q^z, r, a, (\alpha_z))) \quad = \quad \sum_{h=1}^{m_h} \sum_{v=1}^{m_v} f_{(Kv,Kh)} D(1, 1, C(q^z, (r_z), a, (\alpha_z))),$$

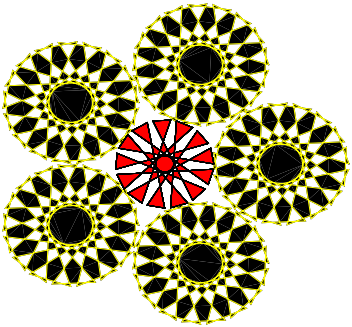
k_0=5, k_1=3, n_0=44, n_1=34



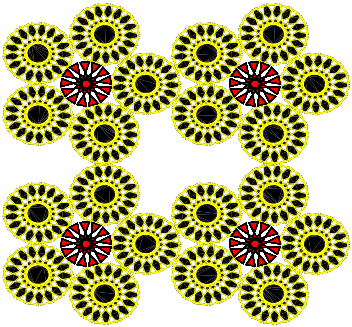
k_0=5, k_1=3, n_0=44, n_1=34



k_0=15, k_1=5; n_0=34, n_1=78

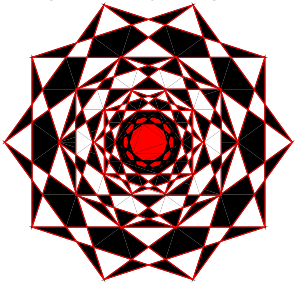


k_0=15, k_1=5; n_0=34, n_1=78

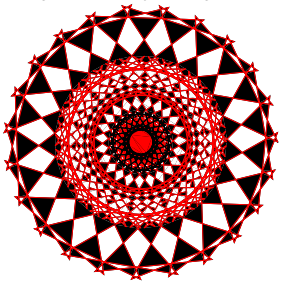


$$D(m_v, m_h, C(q^z, (r_z), (a_z), (\alpha_z))) = \sum_{h=1}^{m_h} \sum_{v=1}^{m_v} f_{(s)} D(1, 1, C(q^z, sr_z, a_z, \alpha_z)), \quad (11)$$

$(k_0,k_1,k_2,k_3)=(1,1,1,1)$; $(n_0,n_1,n_2,n_3)=(20,20,20,20)$

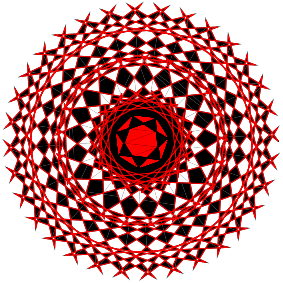


$(k_0,k_1,k_2,k_3)=(5,5,5,5)$; $(n_0,n_1,n_2,n_3)=(42,46,74,46)$

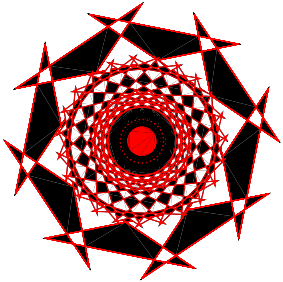


Modeling of Islamic design using overlapped cells

$(k_0,k_1,k_2,k_3)=(3,1,4,3)$; $(n_0,n_1,n_2,n_3)=(42,32,42,64)$

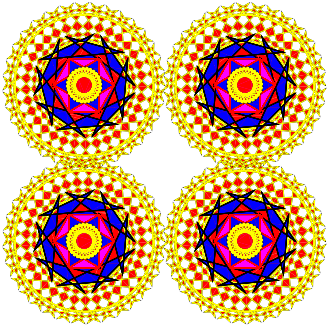


$(k_0,k_1,k_2,k_3)=(5,1,3,4)$; $(n_0,n_1,n_2,n_3)=(66,44,46,64)$

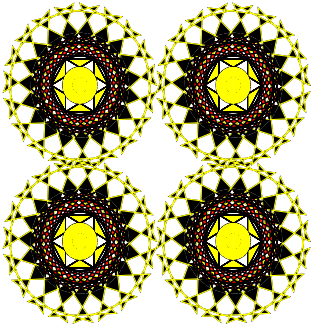


Modeling of Islamic design using overlapped cells

$(k_0,k_1,k_2,k_3)=(5,8,2,5)$; $(n_0,n_1,n_2,n_3)=(56,96,36,66)$

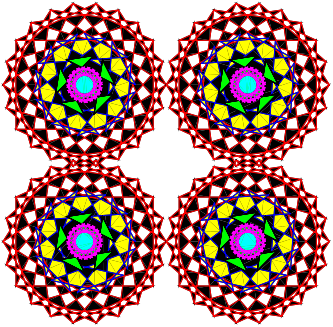


$(k_0,k_1,k_2,k_3)=(10,5,4,4)$; $(n_0,n_1,n_2,n_3)=(118,60,70,38)$

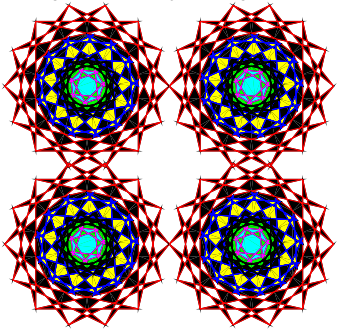


Modeling of Islamic design using overlapped cells

$(k_i,k_1,k_2,k_3)=(3,2,2,3)$; $(n_0,n_1,n_2,n_3)=(40,20,22,44)$



$(k_i,k_1,k_2,k_3)=(1,1,4,1)$; $(n_0,n_1,n_2,n_3)=(14,28,52,28)$



References

S. J. Abas, A. S. Salman, Symmetries of Islamic Geometrical Patterns, World Scientific Publishing Company (1994) Mostafa ZAHRI, Naji Khallaf, Some Results on Riemannean Multiple Barycenters, Journal of Applied Mathematics and Physics, 2014, 2, 503-509. Mostafa ZAHRI, Mathematical Modeling of a Class of Symmetrical Islamic Design, Symmetry 2019, 11, 517. W. Metzler, W. Beau, W. Frees, and A. Ueberla , Symmetry and Self-Similarity with Coupled Logistic Maps,Z. Naturforsch. 42a, 310-318 (1987) P. Chossat and M. Golubitsky, Symmetry increasing bifurcations of chaotic attractors, Physica D 32 (1988) 423â436. C. Humbert, Islamic Ornamental Design, Faber and Faber, Ltd, London, 1980. A.N.J. Th. a Th. van der Hoop, Indonesian Ornamental Design, A.C. Nix and Co., Bandoeng, 1949. John Crossley, The Emergence of Number, Upside Down A Book Company, Yarra Glen, Victoria, Australia, 1980. E. J. Grube, Islamic Pottery, Faber and Faber, Ltd, London, 1976. H. Weyl, Symmetry, Princeton University Press, Princeton, New Jersey, 1952.

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