Numerical Methods for SDEs and SPDEs

Computational Approach

Mostafa Zahri

Department of Mathematics
Taibah University
Al-Madinah Al Munawarah

19. November 2016

Part II

Numerical Schemes for SDEs

- *Itô Taylor-Expansion*
- *Itô Taylor Family Schemes (Strong and Weak convergence)*
- *Analysis of Itô Taylor Schemes*

Langevin Equation

Differential equations with noise Langevin used 'noisy' differential equations to model the particle dynamics of Brownian motion in the early 1900s:

$$\boxed{\frac{dx}{dt} = a(t,x) + b(t,x)\eta_t, \qquad \text{noise} : \eta_t} \tag{1}$$

Gaussian white noise: many mathematical problems finally resolved by Itô in the 1940s with the introduction of stochastic calculus
$\Rightarrow$ stochastic differential equations (SDEs)

$$\boxed{\frac{dx}{dt} = a(t,x) + b(t,x)dW_t, \qquad \text{Wiener process} : W_t} \tag{2}$$

more regular noise $\Rightarrow$ random ordinary differential equations (RODEs)

Brownian motion derivatives: White and colored noises

$Itô - Integral$

$$\frac{1}{2}((\Delta W_{t_i})^2 - \Delta t_i) = \int_{t_i}^{t_{i+1}} W_s dW_s$$

Gaussian (White Noise)

$$'\frac{d}{dt}W'_t = \eta_t$$

$$\eta_t \sim \frac{1}{\sqrt{\Delta t}}\mathcal{N}(0,1)$$

W_t

BM increment

$$\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}$$

$$\Delta W_{t_i} \sim \mathcal{N}(0, \sqrt{\Delta t_i})$$

Gaussian (Random Variable)

$$\int_{t_i}^{t_{i+1}} W_s ds = Z_{\Delta_{t_i}}$$

$$Z_t \sim \sqrt{\frac{t^3}{3}}\mathcal{N}(0,1)$$

The white noise as a stationary process has the following properties:

$$E(\xi_{\Delta t}) = 0; \quad Var(\xi_{\Delta t}) = \frac{1}{\Delta t}; \; Cov\Big(\xi_{\Delta t}(t), \xi_{\Delta s}(s)\Big) = 0; \quad \text{if } t \neq s$$

where $\delta_{\Delta t}(t)$) is an approximation of the following $\delta-$function: This noise is called white whenever one talks about uncorrelated (or independent) noise at each pixel. White noise is the noise signal whose power spectrum is flat (the Fourier transform of its covariance). Otherwise the noise is called Colored Noise.

One-dim Itô coefficient functions

For $Y_t = f(t, X_t)$ consider the one dimensional SDE

$$dY_t = a(t, X_t)dt + b(t, X_t)dW_t$$

The one dimensional Itô Formula is given by

$$\boxed{dY_t = L^0 f(t, X_t)dt + L^1 f(t, X_t)dW_t,} \tag{3}$$

Where the differential operators L^0 and L^1 are given as follows:

$$L^0 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \underbrace{\frac{1}{2}b^2\frac{\partial^2}{\partial x^2}}_{\text{due to } E(\Delta W_t)=\Delta t} \qquad \text{and} \qquad L^1 = b\frac{\partial}{\partial x}.$$

Multi-dim Itô coefficient functions

Given the Itô Formula for $Y_t = f(t, X_t)$:

$$\begin{aligned} dY_t &= \left\{ \frac{\partial f}{\partial t} + \sum_{k=1}^{d} a_t^k \frac{\partial f}{\partial x^k} + \frac{1}{2} \sum_{k,i=1}^{d} \sum_{j=1}^{m} b^{i,j} b^{k,j} \frac{\partial^2 f}{\partial x^i \partial x^k} \right\} dt \\ &+ \sum_{j=1}^{m} \sum_{i=1}^{d} b^{i,j} \frac{\partial f}{\partial x^i} dW_t^j \\ &= L^0 f + \sum_{j=1}^{m} L^j f dW_t^j, \end{aligned}$$

where for $i = 1, \ldots, d$ and $j = 1, \ldots, m$ the Itô coefficient functions are given by

$$\begin{aligned} L^0 &= \frac{\partial}{\partial t} + \sum_{k=1}^{d} a_t^k \frac{\partial}{\partial x^k} + \frac{1}{2} \sum_{k,i=1}^{d} \sum_{j=1}^{m} b^{i,j} b^{k,j} \frac{\partial^2}{\partial x^i \partial x^k}, \\ L^j &= \sum_{i=1}^{d} b^{i,j} \frac{\partial}{\partial x^i}. \end{aligned}$$

One dimensional Itô Taylor Expansion

Consider the one dimensional Itô-Process X_t, which satisfy the Itô-Integral equation:

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)ds + \int_{t_0}^{t} b(X_s)dW_s \tag{4}$$

where a and b reel valued smooth functions, satisfying time-growth condition and $f \in C^2(\mathbb{R}, \mathbb{R})$. By applying the Itô Formula for $f(X_s)$, it follows:

$$\begin{aligned} f(X_s) &= f(X_{t_0}) + \int_{t_0}^{t} \left(a(X_s)\frac{\partial}{\partial x} f(X_s) + \frac{1}{2} b^2(X_s)\frac{\partial^2}{\partial x^2} f(X_s) \right) ds \\ &\quad + \int_{t_0}^{t} b(X_s)\frac{\partial}{\partial x} f(X_s) dW_s \\ &= f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s) ds + \int_{t_0}^{t} L^1 f(X_s) dW_s. \end{aligned}$$

One dimensional Itô Taylor Expansion: Example

If $f(x) = x \in \mathbb{R}$ then $L^0 f = a$ and $L^1 f = b$ the Itô-Taylor Expansion is given in integral form as

$$\begin{aligned} X_t &= X_{t_0} + \int_{t_0}^{t} \left(a(X_{t_0}) + \int_{t_0}^{s} L^0 a(X_z) dz + \int_{t_0}^{s} L^1 a(X_z) dW_z \right) ds \\ &\quad + \int_{t_0}^{t} \left(b(X_{t_0}) + \int_{t_0}^{s} L^0 b(X_z) dz + \int_{t_0}^{s} L^1 b(X_z) dW_z) \right) dW_s \\ &= X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s + \underbrace{R_1}_{Remainder} \end{aligned}$$

where the remainder term is given by:

$$\begin{aligned} R_1 &= \int_{t_0}^{t} \int_{t_0}^{s} L^0 a(X_z) dz ds + \int_{t_0}^{t} \int_{t_0}^{s} L^1 a(X_z) dW_z ds \\ &\quad + \int_{t_0}^{t} \int_{t_0}^{s} L^0 b(X_z) dz dW_s + \int_{t_0}^{t} \int_{t_0}^{s} \underbrace{L^1 b(X_z)}_{Expand\ next} \, dW_z dW_s. \end{aligned}$$

By applying the Itô-Formula to the function $L^1 b$, we get

$$\begin{aligned} X_t &= X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} ds + b(X_{t_0}) \int_{t_0}^{t} dW_s \\ &\quad + L^1 b(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dW_z dW_s + \underbrace{R_2}_{Remainder} \end{aligned}$$

where the remainder term is given by:

$$\begin{aligned} R_2 &= \int_{t_0}^{t} \int_{t_0}^{s} L^0 a(X_z) dz ds + \int_{t_0}^{t} \int_{t_0}^{s} L^1 a(X_z) dW_z ds \\ &\quad + \int_{t_0}^{t} \int_{t_0}^{s} L^0 b(X_z) dz dW_s + \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^0 L^1 b(X_u) du dW_z dW_s \\ &\quad + \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^1 \underbrace{L^1 b(X_u)}_{Expand\ next} dW_u dW_z dW_s. \end{aligned}$$

Don't forget $dtdt = dtdW_t = dW_t dt = 0$ and $dW_t dW_t = dt$.

Multi-Index: Hierarchical and Remainder Index-Sets

A set $\mathcal{A}$ of Multi-Index is called **Hierarchical Index-Set**, if the following condition are satisfied

- $\mathcal{A} \neq \varnothing$,
- $\ell_m = \sup_{\alpha \in \mathcal{A}} \quad l(\alpha) < \infty$,
- $-\alpha \in \mathcal{A}; \quad \forall\ \alpha \in \mathcal{A} \backslash \{v\}$.

For any hierarchical set $\mathcal{A}$, we define the corresponding **Remainder Index-Set** as:

$$\mathcal{B}(\mathcal{A}) = \{\alpha \in \mathcal{M} \backslash \{\mathcal{A}\} : -\alpha \in \mathcal{A}\}.$$

General Form of the Itô-Taylor-Expansion

Theorem

Let ρ and τ are two Stop-times such that $t_0 \leqslant \rho(w) \leqslant \tau(w) \leqslant T$. If the function $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}$ and all its differential are smooth and if the partial differential of a and b exist, then for any set $\mathcal{A} \in \mathcal{M}$ follow the stochastic Taylor-Expansion

$$f(\tau, X_\tau) = \sum_{\alpha \in \mathcal{A}_\gamma} I_\alpha[f_\alpha(\rho, X_\rho)] + \sum_{\alpha \in \mathcal{B}(\mathcal{A}_\gamma)} I_\alpha[f_\alpha(., X.)]_{\rho,\tau}, \tag{5}$$

Where the **Itô coefficient functions** are defined by

$$f_\alpha = \begin{cases} f & \text{if} \quad \ell(\alpha) = 0; \\ L^{j_1} f_{-\alpha} & \text{if} \quad \ell(\alpha) > 0. \end{cases}$$

for $\alpha = (j_1, \ldots, j_l) \in M$ and $f \in C^{n(\alpha), \ell(\alpha)+n(\alpha)}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R})$.

Convergence Theorem: Source of the convergence order

For $l \in \mathbb{N}$ and $\alpha = (j_1, \ldots, j_l)$ it exists a positive constant K independent from Δ such that

$$\boxed{\left(I\!E(I^2_{\alpha,\Delta})\right)^{\frac{1}{2}} \leqslant K\Delta^{\frac{\ell(\alpha)+n(\alpha)}{2}}} \tag{6}$$

Moreover, for X a d-dim stoch. process sol. of SDE, a multi-Index $\alpha \in \mathcal{M}$ and $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}$ satisfying If $|f_\alpha(t,x)|^2 \leqslant K\,(1 + |x|^2)$, the we have

$$\boxed{I\!E\left|I_\alpha\left[f_\alpha(t, X_t)\right]_{0,\Delta}\right|^2 \leqslant K\left(1 + I\!E|X_t|^2\right)\Delta^{l(\alpha)+n(\alpha)}} \tag{7}$$

Remark that the real nuber $\frac{\ell(\alpha)+n(\alpha)}{2} = \gamma$ could be used as a convergence order of the Itô-Taylor approximation of SDE solutions.

So, we have to characterize the sets $\mathcal{A}_\gamma$ and $\mathcal{B}_\gamma(\mathcal{A}_\gamma)$.

Convergence Theorem, cont.

Theorem

Consider the strong Itô-Taylor approximation $Y^{\Delta} = \left\{Y_t^{\Delta},\ t \in]t_0, T[\right\}$ *with convergence order* $\gamma = 0.5, 1, 1.5, \ldots$. *If*

- $I\!E(|X_{t_0}|^2) < \infty$ *and* $\left(I\!E(|X_{t_0} - Y_{t_0}^{\Delta}|^2)\right)^{\frac{1}{2}} \leqslant K_1 \Delta^{\gamma}$
- $|f_\alpha(t,x) - f_\alpha(t,y)| \leqslant K_2|x-y| \qquad \forall \alpha \in \mathcal{A}_\gamma,\ t \in [t_0, T]$ *and* $x, y \in \mathbb{R}^d$,
- $f_{-\alpha} \in C^{1,2}$ *and* $f_\alpha \in H_\alpha \qquad \forall \alpha \in \mathcal{A}_\gamma \cup \mathcal{B}_\gamma$,
- $|f_\alpha(t,x)| \leqslant K_3(1+|x|) \qquad \forall \alpha \in \mathcal{A}_\gamma \cup \mathcal{B}_\gamma, t \in [t_0, T]$ *and* $x \in \mathbb{R}^d$.

then it holds

$$I\!E\left(|X_T - Y_T^{\Delta}|\right) \leqslant K_4 \Delta^{\gamma}. \tag{8}$$

where $K_i, i = 1, 2, 3, 4.$ *are real pairwise independent positive constants .*

For the proof we refer to [2] p. 361ff)

On the Convergence Theorem Assumptions

The convergence theorem assumes that

- the coefficient functions f_α in the Taylor scheme are uniformly bounded on $\mathbb{R}^d$: i.e., the partial derivatives of appropriately high order of the SDE coefficient functions $a, b^1, \ldots, b^m$ are uniformly bounded on R^d.
- However, this assumption is **not satisfied in many important applications**:

1. **Cox-Ingersoll-Ross models** in finance with $V_t \in [0, 1]$

$$dV_t = \alpha(\beta - V_t)dt + \mu\sqrt{V_t}dW_t$$

2. **Duffing-van der Pol oscillator** with multiplicative noise:

$$\begin{aligned} dX_t^1 &= X_t^2 dt \\ dX_t^2 &= \Big(- X_t^1 + \beta X_t^2 - (X_t^1)^3 - (X^1 t)^2 X^2 t \Big) dt + \sigma X_t^2 dW_t \end{aligned}$$

Proofs in the literature (e.g. in the monographs) see [2, 4]

Construction of Multi-Index Sets in the Itô-Taylor-Expansion

For $\gamma = 0.5, 1, 1.5, 2, \ldots$ the set of Multi-indices $\mathcal{A}_\gamma$ is called Hierarchical Index-Set if

$$\mathcal{A}_\gamma = \{\alpha \in \mathcal{M} : \ \ell(\alpha) + n(\alpha) \leqslant 2\gamma \ or \ \ell(\alpha) = n(\alpha) = \gamma + \frac{1}{2}\}.$$

- Remark that the three assumption of a Hierarchical set are satisfied.
- The construction of the corresponding remainder set is given by

$$\mathcal{B}(\mathcal{A}_\gamma) = \{\alpha \in \mathcal{M} \backslash \mathcal{A}_\gamma : -\alpha \in \mathcal{A}_\gamma\}.$$

- (Later) γ will refer the convergence order the Itô-Taylor family schemes.

Strong and Weak convergence

In the EM scheme, the numerical solution to the SDE, Y_t converges to the exact solution X_t. We have to distinct between the following types of convergence :

- (**Path Approximation**). We said that the **convergence is strong** of order γ if
$$\lim_{N_T \to \infty} Es(\gamma) := \lim_{N_T \to \infty} E\Big(|Y^{\gamma}_{N_T} - X_t|\Big) = 0.$$
- (**Distributional Approximation**). We said that the **convergence is weak** of order γ if
$$\lim_{N_T \to \infty} Ew(\gamma) := \lim_{N_T \to \infty} \Big|Eg\Big(Y^{\gamma}_{N_T}\Big) - Eg(X_t)\Big| = 0.$$

Where N_T refers the number of the time steps used for the approximation, g any polynomial function.

Examples of Hierarchical and Remainder Index-Sets

For $F = \{0, 1\}$, we have

- If $\ell_m = 0$ (i.e $\gamma = 0$) then $\mathcal{A}_0 = \{v\}$ and $\mathcal{B}(A_0) = \{(0), (1)\}$.
- If $\ell_m = 1$ (i.e $\gamma = 0.5$) then $\mathcal{A}_{0.5} = \{v, (0), (1)\}$ and

$$\mathcal{B}(\mathcal{A}_{0.5}) = \{(0,0), (1,0), (0,1), (1,1)\}.$$

- If $\ell_m = 2$ (i.e $\gamma = 1$) then $\mathcal{A}_1 = \{v, (0), (1), (1,1)\}$ and

$$\mathcal{B}(\mathcal{A}_1) = \{(0,0), (1,0), (0,1), (0,1,1), (1,1,1)\}.$$

- If $\ell_m = 3$ (i.e $\gamma = 1.5$) then $\mathcal{A}_{1.5} = \{v, (0), (1), (1,1), (0,0), (1,0), (0,1), (1,1,1)\}$ and $\mathcal{B}(\mathcal{A}_{1.5}) =$
$\{(0,0,0), (1,0,0), (1,1,0), (0,1,0), (1,0,1), (0,0,1), (0,1,1), (0,1,1,1), (1,1,1,1)\}$.

Hierarchical Index-Sets, cont.

$$\mathcal{A}_{0.0} = \{v\}$$

$\gamma = 0.5$ Euler-Maruyama

$$\mathcal{A}_{0.5} = \{v, (0), (1)\}$$

$\gamma = 1.$ Milstein/ Taylor

$$\mathcal{A}_{1.0} = \{v, (0), (1), (1,1)\}$$

$\gamma = 1.5$ Taylor

$$\mathcal{A}_{1.5} = \{v, (0), (1), (1,1), (0,0), (1,0), (0,1), (1,1,1)\}$$

$\gamma = 2$ Taylor

$$\mathcal{A}_{2.0} = \Big\{v, (0), (1), (1,1), (0,0), (1,0), (0,1), (1,1,1),$$
$$(1,1,0), (0,1,1), (1,0,1), (0,1,1), (1,1,1,1)\Big\}$$

Itô-Taylor Family Schemes: Strong Convergence

The strong Itô-Taylor scheme of order $\gamma = 0.5,\ 1,\ 1.5,\ 2, \ldots$ is given by

$$\boxed{Y_{n+1} = Y_n + \sum_{\alpha \in \mathcal{A}_\gamma \setminus \{v\}} I_\alpha[f_\alpha(t_n, Y_n)]_{t_n, t_{n+1}}} \tag{9}$$

for $n = 0, 1, 2, \ldots$, and the initial value $Y_0 = \xi_0$.
$\mathcal{M} = \{v\} \bigcup \{\alpha = (j_1, j_2, \ldots, j_l); j_i \in F;\ l \in \{1, 2, \ldots\}\}$.

Example of Itô-Taylor scheme order 1.5

Consider the autonomous case $dX_t = a(X_t)dt + b(X_t)dW_t$.
An example of Itô-Taylor scheme for a d dimensional Itô-Process X_t and an index set $\mathcal{M} = \{1, \dots, m\}$ (i.e m dimensional BM) is given by:

$$
\begin{aligned}
Y_{n+1} &= Y_n + I_v[a(Y_n)]_{t_n,t_{n+1}} + \sum_{j=1}^{m} I_{(j)}[b^j(Y_n)]_{t_n,t_{n+1}} && (10)\\
&+ \sum_{j_1,j_2=1}^{m} I_{(j_1,j_2)}[L^{j_1}b^{j_2}(Y_n)]_{t_n,t_{n+1}} && (11)\\
&+ \sum_{j=1}^{m} I_{(j,0)}[L^j a(Y_n)]_{t_n,t_{n+1}} + \sum_{j=1}^{m} I_{(0,j)}[L^0 b^j(Y_n)]_{t_n,t_{n+1}} && (12)\\
&+ \sum_{j=1}^{m} I_{(0,0)}[a(Y_n)]_{t_n,t_{n+1}} + \sum_{j,j_1,j_2=1}^{m} I_{(j,j_1,j_2)}[L^j L^{j_1} b^{j_2}(Y_n)]_{t_n,t_{n+1}} && (13)
\end{aligned}
$$

Euler Maruyama (10), Milstein (10)-(11) and Taylor of order 1.5 is represented by (10)-(13).

The one-dimensional Itô-Taylor $\gamma = 1.5$, $(d = m = 1)$

$$
\begin{aligned}
I_v[a(X_n)]_{t_n,t_{n+1}} &= a(X_n)\Delta t_n \\
I_{(1)}[b^1(X_n)]_{t_n,t_{n+1}} &= b^1(X_n)\int_{t_n}^{t_{n+1}} dW_t^1 \\
&= b(X_n)\Delta W_{t_n} \\
I_{(1,1)}[L^1 b^1(t_n,X_n)]_{t_n,t_{n+1}} &= b^1\frac{\partial b^1}{\partial x}(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW_s^1 dW_t^1 \\
&= b(X_n)\frac{\partial b}{\partial x}(X_n)\frac{1}{2}\left((\Delta W_{t_n})^2 - \Delta t_n\right) \\
I_{(1,0)}[L^1 a(X_n)]_{t_n,t_{n+1}} &= b^1(X_n)\frac{\partial a}{\partial x}(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW_s^1 dt \\
&= b(X_n)\frac{\partial a}{\partial x}(X_n)\Delta Z_{t_n} \qquad \Delta Z_{t_n} \sim N\left(0, \frac{1}{3}(\Delta t_n)^3\right)
\end{aligned}
$$

Recall $\quad L^0 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \frac{1}{2}b^2\frac{\partial}{\partial x^2}, \qquad L^1 = b\frac{\partial}{\partial x}.$

The 1-dim case Itô-Taylor $\gamma = 1.5$, $(d = m = 1)$ cont.

$$
\begin{aligned}
I_{(0,1)}[L^0 b^1(X_n)]_{t_n,t_{n+1}} &= \left(a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\right)(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} ds dW_t \\
&= \left(a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\right)(X_n)\left(\Delta t_n \Delta W_{t_n} - \Delta Z_{t_n}\right) \\
I_{(0,0)}[L^0 a(X_n)]_{t_n,t_{n+1}} &= L^0 a(X_n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} ds dt \\
&= \left(a\frac{\partial a}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 a}{\partial x^2}\right)(X_n)\frac{1}{2}(\Delta t_n)^2 \\
I_{(1,1,1)}[L^1 L^1 b^1(X_n)]_{t_n,t_{n+1}} &= L^1 L^1 b\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\int_{t_n}^{s} dW_r dW_s dW_t \\
&= \left(b\left(\frac{\partial b}{\partial x}\right)^2 + b^2\frac{\partial^2 b}{\partial x^2}\right)(X_n)\left(\frac{(\Delta W_{t_n})^2}{6} - \frac{\Delta t_n}{2}\right)\Delta W_{t_n}
\end{aligned}
$$

Strong Euler-Maruyama Scheme (1955), $\gamma = 0.5$

Consider the **Geometric Brownian Motion**, which satisfies the SDE

$$\boxed{\begin{aligned} X_0 &= \xi \\ dX_t &= \lambda X_t dt + \mu X_t dW_t \end{aligned}}$$

where λ is called the percentage drift and μ is called the percentage volatility. The exact solution is

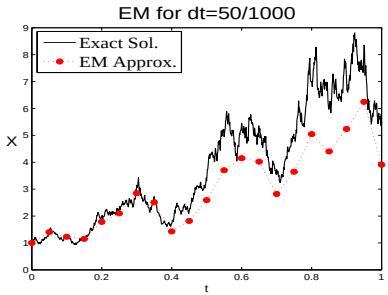
$$X_t = X_0 \exp\left((\lambda - \frac{\mu^2}{2})t - \mu W_t\right)$$

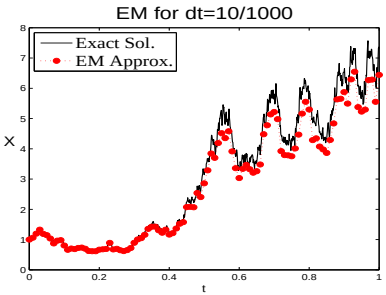
Numerical solution: for $\mathcal{A}_{0.5} = \{v, (0), (1)\}$ (i.e $\ell_1 = 1$), we define the **Euler-Maruyama** scheme

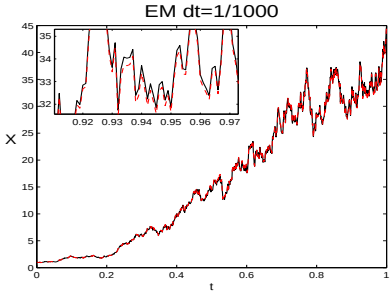
$$\boxed{\begin{aligned} X_0 &= Y_0 = \xi \\ Y_{n+1} &= Y_n + \lambda Y_n \Delta t_n + \mu Y_n \Delta W_{t_n} \end{aligned}}$$

Where Y_n in the numerical approximation of the exact solution X_{t_n}.

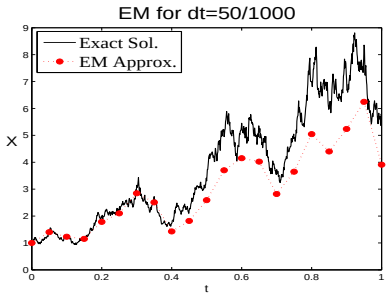
Numerical Simulation of GBM by using EM.

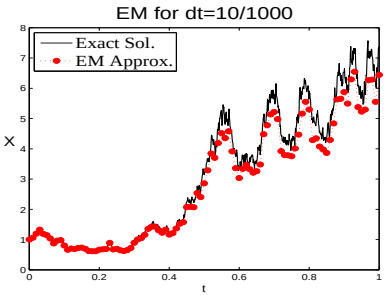


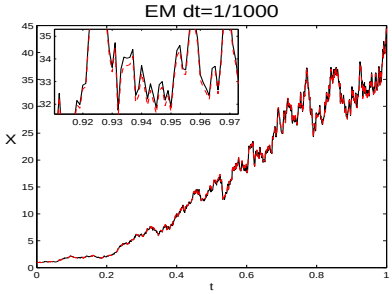


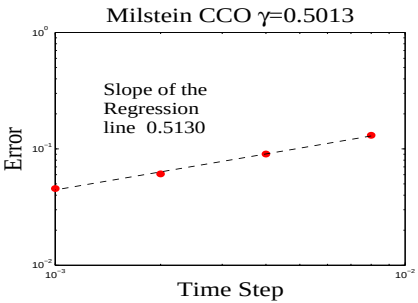


Numerical Simulation of GBM by using EM.









Where $dt = 10^{-3}$ is used to simulate exact solution.

General form of EM scheme

Consider the SDE

$$\boxed{\begin{aligned} X_0 &= \xi \\ dX_t &= a(t, X_t)dt + b(t, X_t)dW_t \end{aligned}}$$

Numerical solution: for $\mathcal{A}_{0.5} = \{v, (0), (1)\}$ (i.e $\ell_1 = 1$), we define the **Euler-Maruyama** scheme

$$\boxed{\begin{aligned} X_0 = Y_0 &= \xi \\ Y_{n+1} &= Y_n + a(t_n, Y_n)\Delta t_n + b(t_n, Y_n)\Delta W_{t_n} \end{aligned}}$$

Where Y_n in the numerical approximation of the exact solution X_{t_n} and

$$\begin{aligned} \Delta t_n &= \int_{t_n}^{t_{n+1}} dt \\ \Delta W_{t_n} &= \int_{t_n}^{t_{n+1}} dW_s. \end{aligned}$$

Stochastic Milstein Scheme (1974), $\gamma = 1$

For $\mathcal{A}_1 = \{v, (0), (1), (1,1)\}$ (i.e $\ell_1 = 2$), we define the **Milstein** scheme

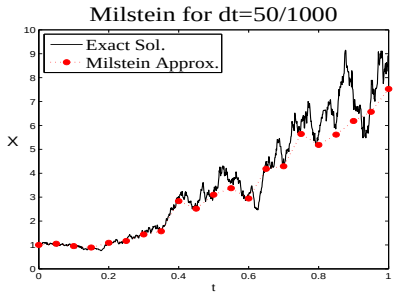
$$\boxed{\begin{aligned} X_0 &= Y_0 = \xi \\ Y_{n+1} &= Y_n + a(t_n, Y_n)\Delta t_n + b(t_n, Y_n)\Delta W_{t_n} \\ &\quad + \frac{1}{2} b(t_n, Y_n)\frac{\partial b}{\partial x}(t_n, Y_n))((\Delta W_{t_n})^2 - \Delta t_n) \end{aligned}} \tag{14}$$

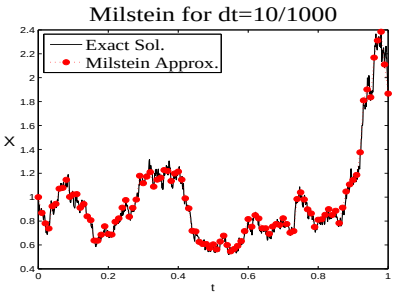
Where Y_n in the numerical approximation of the exact solution X_{t_n} and the additional (to E.M) is

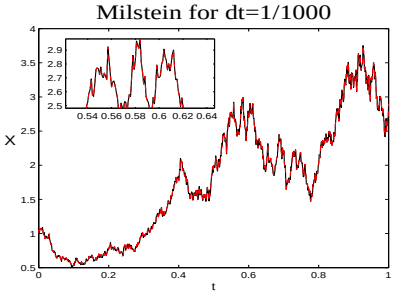
$$f_{(1,1)} I_{(1,1)} \quad = \quad L^1 b(t_n, Y_n) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s^1 dW_t^1.$$

The Milstein method is a first order Itô-Taylor method. It is derived from a truncation of the stochastic Itô-Taylor expansion of the solution.

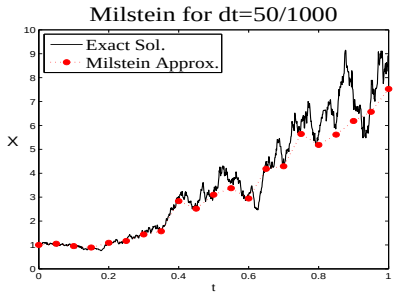
Numerical Simulation of GBM by using Milstein.

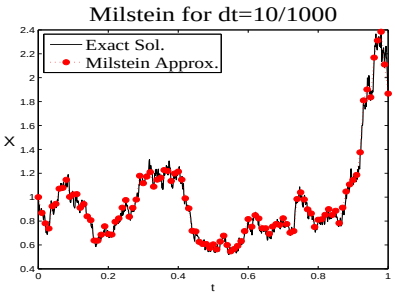


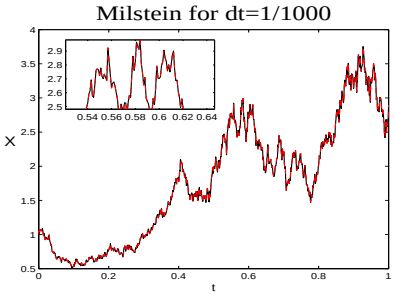


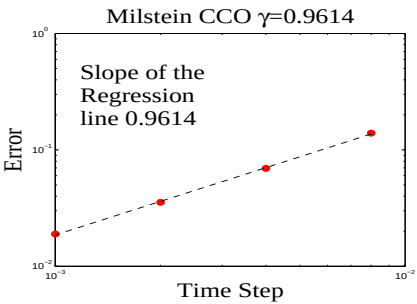


Numerical Simulation of GBM by using Milstein.









Where $dt = 10^{-3}$ is used to simulate exact solution.

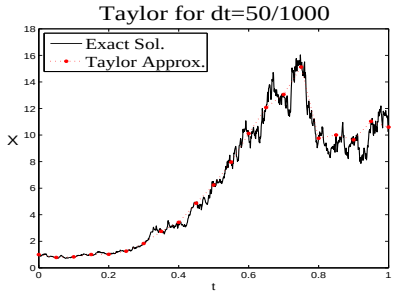
Stochastic Taylor Scheme with order $\gamma = 1.5$

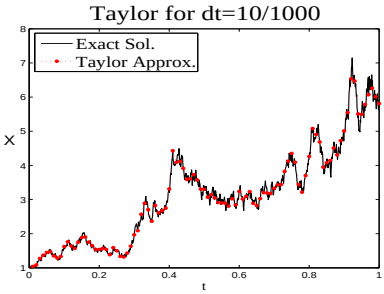
For $\mathcal{A}_{1.5} = \{v, (0), (1), (1,1), (0,0), (1,0), (0,1), (1,1,1)\}$ (i.e $\ell_1 = 3$), we define the **Itô-Taylor** scheme with order $\gamma = 1.5$:

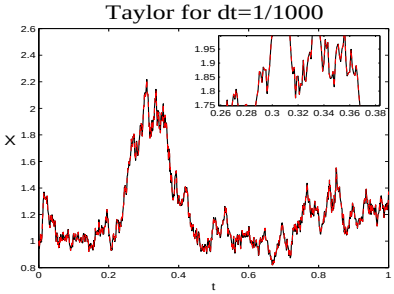
$$
\begin{aligned}
X_0 &= Y_0 = \xi \\
Y_{n+1} &= Y_n + a\Delta t_n + b\Delta W_{t_n} + \frac{1}{2} b \frac{\partial b}{\partial x}((\Delta W_{t_n})^2 - \Delta t_n) \\
&\quad + \frac{1}{2}\left(a\frac{\partial a}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 a}{\partial x^2}\right)(\Delta t)^2 + b\frac{\partial a}{\partial x}\Delta Z_{t_n} \\
&\quad + \left(a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\right)\left(\Delta t_n \Delta W_{t_n} - \Delta Z_{t_n}\right) \\
&\quad + \frac{1}{2}\left(b\left(\frac{\partial b}{\partial x}\right)^2 + b^2\frac{\partial^2 b}{\partial x^2}\right)\left(\frac{1}{3}(\Delta W_{t_n})^2 - \Delta t_n\right)\Delta W_{t_n}
\end{aligned}
\tag{15}
$$

Where Y_n in the numerical approximation of the exact solution X_{t_n} and ΔZ_{t_n} is a normal distributed random variable $\Delta Z_{t_n} \sim N(0, \frac{1}{3}(\Delta t_n)^3)$

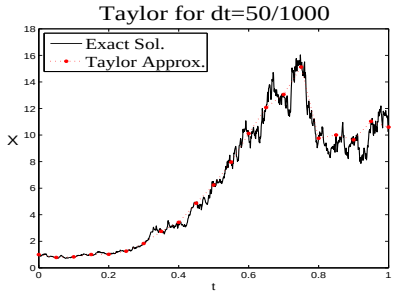
Numerical Simulation of GBM by using Taylor.

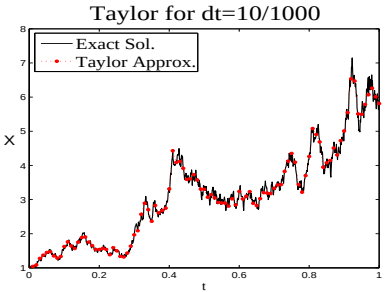


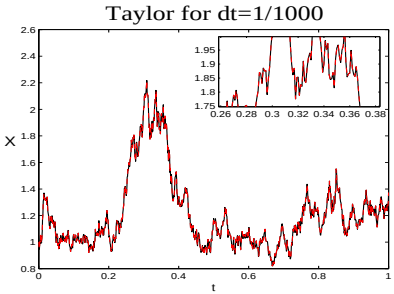


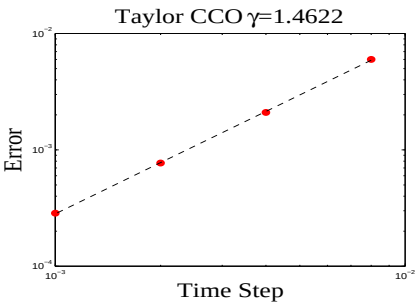


Numerical Simulation of GBM by using Taylor.



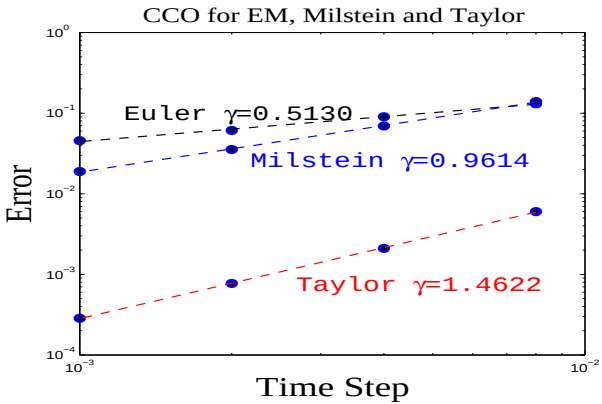






Where $dt = 10^{-3}$ is used to simulate exact solution.

Computational Convergence Orders



Increasing the number of discretization time-points and the number of simulations leads to the best estimation of computational convergence orders. In our cases, we have used a time step $dt = 10^{-3}$ and $M = 1000$ simulation for each numerical simulation.

Strong first order Runge-Kutta method

In the Milstein scheme, the replacement of the first derivative of the diffusion b_x by

$$\frac{\partial b}{\partial x}(X_{t_n}) \approx \frac{b\Big(X_{t_n} + b(X_{t_n})\sqrt{\Delta t_n}\Big) - b\Big(X_{t_n}\Big)}{b\Big(X_{t_n}\Big)\sqrt{\Delta t_n}}$$

leads to Strong Order 1.0 **Runge-Kutta** Method

$$\begin{aligned} X_0 &= Y_0 = \xi \\ Y_{n+1} &= Y_n + a(Y_n)\Delta t_n + b(Y_n)\Delta W_{t_n} \\ &\quad + \frac{1}{2}\left(b\Big(Y_n + b(Y_n)\sqrt{\Delta t_n}\Big) - b\Big(Y_n\Big)\right)\frac{(\Delta W_{t_n})^2 - \Delta t_n)}{\sqrt{\Delta t_n}} \end{aligned} \tag{16}$$

Weak approximation of SDE

For $\beta = 1, 2, \ldots$ the set of Multi-indices $\mathcal{A}_\beta^w$ is called Hierarchical Index-Set for the **weak approximation** if

$$\mathcal{A}_\beta^w = \{\alpha \in \mathcal{M} : \ \ell(\alpha) \leqslant \beta\}.$$

The weak approximation of the solution of a SDE is given by

$$\boxed{\begin{aligned} X_0 &= Y_0 = \xi \\ Y_n &= \sum_{\alpha \in \mathcal{A}_\beta^w} I_\alpha[f_\alpha(t_n, Y_{t_n})] \end{aligned}} \tag{17}$$

Convergence with weak order β requires the regularity of the drift vector-function a and the regularity of the entries of the diffusion matrix b. This is necessary to ensure that f_α are Lipschitz and have bounded linear growth.

Weak Taylor Schemes, $\beta = 1$ and $\beta = 2$

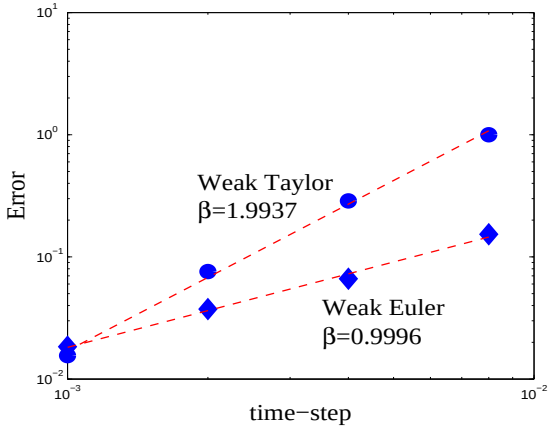
For $\mathcal{A}_1 = \{v, (0), (1)\}$ (i.e $\ell_1 = 1$) (the same as by the strong scheme), we define the **Weak Euler-Maruyama** scheme with order $\beta = 1$

$$\boxed{\begin{aligned} X_0 &= Y_0 = \xi \\ Y_{n+1} &= Y_n + a(Y_n)\Delta t_n + b(Y_n)\Delta W_{t_n} \end{aligned}} \tag{18}$$

Where Y_n in the numerical approximation of the exact solution X_{t_n}.
For $\mathcal{A}_2 = \{v, (0), (1), (0,0), (0,1), (1,0), (1,1)\}$ (i.e $\ell_1 = 2$), we define the **Weak Taylor** with order $\beta = 2$

$$\boxed{\begin{aligned} X_0 &= Y_0 = \xi \\ Y_{n+1} &= Y_n + a\Delta t_n + b\Delta W_{t_n} + \frac{1}{2} b \frac{\partial b}{\partial x}((\Delta W_{t_n})^2 - \Delta t_n) \\ &\quad + \frac{1}{2}\Big(a\frac{\partial a}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 a}{\partial x^2}\Big)(\Delta t)^2 + b\frac{\partial a}{\partial x}\Delta Z_{t_n} \\ &\quad + \Big(a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2}\Big)\Big(\Delta t_n \Delta W_{t_n} - \Delta Z_{t_n}\Big) \end{aligned}} \tag{19}$$

CCO for WEM and WT



Numerical Sol. of two-dim. Systems of SDE

Given the two dimensional system of SDE

$$dX_t^k = a^k(t, X_t)dt + \sum_{j=1}^{2} b^{k,j}(t, X_t)dW_t^j, \qquad X_0^k = \xi^k \tag{20}$$

For $F = \{0, 1, 2\}$ and $\mathcal{A}_1 = \{v, (0), (1), (2), (1,1), (1,2), (2,2), (2,1)\}$, we define the **two dimensional Milstein** scheme as

$$\begin{aligned} Y_{n+1}^1 &= Y_n^1 + a^1\Delta t_n + b^{1,1}\Delta W_{t_n}^1 + b^{1,2}\Delta W_{t_n}^2 \\ &+ \left(b^{1,1}\frac{\partial b^{1,1}}{\partial x^1} + b^{2,1}\frac{\partial b^{1,1}}{\partial x^2}\right) I_{(1,1)} + \left(b^{1,1}\frac{\partial b^{1,2}}{\partial x^1} + b^{2,1}\frac{\partial b^{1,2}}{\partial x^2}\right) I_{(1,2)} \\ &+ \left(b^{1,2}\frac{\partial b^{1,1}}{\partial x^1} + b^{2,2}\frac{\partial b^{1,1}}{\partial x^2}\right) I_{(2,1)} + \left(b^{1,2}\frac{\partial b^{1,2}}{\partial x^1} + b^{2,2}\frac{\partial b^{1,2}}{\partial x^2}\right) I_{(2,2)}, \end{aligned}$$

$$\begin{aligned} Y_{n+1}^2 &= Y_n^2 + a^2\Delta t_n + b^{2,1}\Delta W_{t_n}^1 + b^{2,2}\Delta W_{t_n}^2 \\ &+ \left(b^{1,1}\frac{\partial b^{2,1}}{\partial x^1} + b^{2,1}\frac{\partial b^{2,1}}{\partial x^2}\right) I_{(1,1)} + \left(b^{1,1}\frac{\partial b^{2,2}}{\partial x^1} + b^{2,1}\frac{\partial b^{2,2}}{\partial x^2}\right) I_{(1,2)} \\ &+ \left(b^{1,2}\frac{\partial b^{2,1}}{\partial x^1} + b^{2,2}\frac{\partial b^{2,1}}{\partial x^2}\right) I_{(2,1)} + \left(b^{1,2}\frac{\partial b^{2,2}}{\partial x^1} + b^{2,2}\frac{\partial b^{2,2}}{\partial x^2}\right) I_{(2,2)} \end{aligned}$$

Computation of double Itô Integrals $I_{(j_1,j_2),\Delta t_n}$

For the numerical computation of $I_{(j_1,j_2),\Delta t_n}$, we have to distinct:

Case 1. $(j_1 = j_2 = j)$

$$I_{(j,j),\Delta t_n} = \frac{1}{2}(I^2_{(j),\Delta t} - \Delta t_n) = \frac{1}{2}\Big((\Delta W^j_{t_n})^2 - \Delta t_n\Big). \tag{21}$$

Case 2. $(j_1 \neq j_2)$

$$\begin{aligned} I^p_{(j_1,j_2),\Delta t_n} &= \frac{1}{2}\Delta t_n R_1 R_2 + \Delta t_n \sqrt{\rho_p}(R_3 R_2 - R_4 R_1) \\ &+ \frac{\Delta t_n}{2\pi}\sum_{r=1}^{p}\frac{1}{r}(R_5(\sqrt{2}R_2 + R_8) - R_6(\sqrt{2}R_1 + R_7)), \end{aligned} \tag{22}$$

where R_i for $i = 1, \ldots, 8$ (i.i.d) are standard normal distributed random variables and pairwise independent. The coefficient ρ_p could be approximated by

$$\rho_p = \frac{1}{2\pi^2}\sum_{r=p+1}^{\infty}\frac{1}{r^2}.$$

Milstein for Systems of SDE, Simplified Notation

In order to be able to define the **Milstein** for a system of differential equation with n equation, we use the following notation

$$\begin{aligned}
\tilde{L}^1 &= \left(b^{1,1}\frac{\partial}{\partial x^1}, b^{2,1}\frac{\partial}{\partial x^2}\right);\\
\tilde{L}^2 &= \left(b^{1,2}\frac{\partial}{\partial x^1}, b^{2,2}\frac{\partial}{\partial x^2}\right);\\
&\quad \text{for} \quad i,j=1,2\\
\tilde{b}^{j,\cdot} &= \left(b^{j,1}, b^{j,2}\right);\ \tilde{b}^{\cdot,j} = \left(b^{1,j}, b^{2,j}\right);\ \tilde{b}^{i,j} = \left(b^{i,j}, b^{i,j}\right)\\
\Delta\tilde{W}_{t_n} &= \left(\Delta W^1_{t_n}, \Delta W^2_{t_n}\right)^t.
\end{aligned}$$

Thus the two dimensional Milstein scheme is

$$\begin{aligned}
Y^1_{n+1} &= Y^1_n + a^1\Delta t_n + \tilde{b}^{1,\cdot}\Delta\tilde{W}_{t_n}\\
&\quad +\tilde{L}^1\tilde{b}^{1,1}I_{(1,1)} + \tilde{L}^1\tilde{b}^{1,2}I_{(1,2)}\\
&\quad +\tilde{L}^2\tilde{b}^{1,1}I_{(2,1)} + \tilde{L}^2\tilde{b}^{1,2}I_{(2,2)}.\\
Y^2_{n+1} &= Y^2_n + a^2\Delta t_n + \tilde{b}^{2,\cdot}\Delta\tilde{W}_{t_n}\\
&\quad +\tilde{L}^1\tilde{b}^{2,1}I_{(1,1)} + \tilde{L}^1\tilde{b}^{2,2}I_{(1,2)}\\
&\quad +\tilde{L}^2\tilde{b}^{2,1}I_{(2,1)} + \tilde{L}^2\tilde{b}^{2,2}I_{(2,2)}.
\end{aligned}$$

Milstein for Systems of SDE, cont.

the **three dimensional Milstein** scheme as

$$
\begin{aligned}
Y^1_{n+1} &= Y^1_n + a^1\Delta t_n + \tilde{b}^{1,\cdot}\Delta\tilde{W}_{t_n} \\
&\quad + \tilde{L}^1\tilde{b}^{1,1}I_{(1,1)} + \tilde{L}^1\tilde{b}^{1,2}I_{(1,2)} + \tilde{L}^1\tilde{b}^{1,3}I_{(1,3)} \\
&\quad + \tilde{L}^2\tilde{b}^{1,1}I_{(2,1)} + \tilde{L}^2\tilde{b}^{1,2}I_{(2,2)} + \tilde{L}^2\tilde{b}^{1,3}I_{(2,3)} \\
&\quad + \tilde{L}^3\tilde{b}^{1,1}I_{(3,1)} + \tilde{L}^3\tilde{b}^{1,2}I_{(3,2)} + \tilde{L}^3\tilde{b}^{1,3}I_{(3,3)} \\
Y^2_{n+1} &= Y^2_n + a^2\Delta t_n + \tilde{b}^{2,\cdot}\Delta\tilde{W}_{t_n} \\
&\quad + \tilde{L}^1\tilde{b}^{2,1}I_{(1,1)} + \tilde{L}^1\tilde{b}^{2,2}I_{(1,2)} + \tilde{L}^1\tilde{b}^{2,3}I_{(1,3)} \\
&\quad + \tilde{L}^2\tilde{b}^{2,1}I_{(2,1)} + \tilde{L}^2\tilde{b}^{2,2}I_{(2,2)} + \tilde{L}^2\tilde{b}^{2,3}I_{(2,3)} \\
&\quad + \tilde{L}^3\tilde{b}^{2,1}I_{(3,1)} + \tilde{L}^3\tilde{b}^{2,2}I_{(3,2)} + \tilde{L}^3\tilde{b}^{2,3}I_{(3,3)} \\
Y^3_{n+1} &= Y^3_n + a^3\Delta t_n + \tilde{b}^{3,\cdot}\Delta\tilde{W}_{t_n} \\
&\quad + \tilde{L}^1\tilde{b}^{3,1}I_{(1,1)} + \tilde{L}^1\tilde{b}^{3,2}I_{(1,2)} + \tilde{L}^1\tilde{b}^{3,3}I_{(1,3)} \\
&\quad + \tilde{L}^2\tilde{b}^{3,1}I_{(2,1)} + \tilde{L}^2\tilde{b}^{3,2}I_{(2,2)} + \tilde{L}^2\tilde{b}^{3,3}I_{(2,3)} \\
&\quad + \tilde{L}^3\tilde{b}^{3,1}I_{(3,1)} + \tilde{L}^3\tilde{b}^{3,2}I_{(3,2)} + \tilde{L}^3\tilde{b}^{3,3}I_{(3,3)}.
\end{aligned}
$$

The deterministic Lorenz system

Lorenz simplified some of the Navier-Stokes equations in the area of fluid dynamics, and obtained the following non-linear three dimensional ordinary differential system:

$$\begin{aligned} \partial_t x &= p(y(t) - x(t)), \\ \partial_t y &= rx(t) - y(t) - x(t)z(t), \\ \partial_t z &= x(t)y(t) - cz(t), \end{aligned} \tag{23}$$

where

- the parameter p is the Prandtl number in a dimensionless parameter of a convecting system that characterizes the regime of convection
- $r = \frac{Ra}{Ra_c}$ is the quotient of the Rayleigh number and critical Rayleigh number
- b is a geometric factor
- Lorenz used the values $p = 10, r = 28, c = \frac{8}{3}$.

The Stochastic Lorenz system

The stochastic differential lorenz system (SDE) given as:

$$d\mathbf{X}_t = a(t, \mathbf{X}_t)dt + b(t, \mathbf{X}_t)d\mathbf{W}_t \qquad \mathbf{X}_0 = \xi, \tag{24}$$

where

- $a : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ represents the drift vector,
- $b : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^{m \times d}$ the diffusion matrix
- $(\mathbf{X}_t)_{t \in [t_0, T]}$ is a $d-$dim reel valued stochastic process solution of SDE (**??**).
- $(\mathbf{W}_t)_{t \in [t_0, T]}$ is a three dimensional Brownian motion.

A stochastic Lorenz system

We numerically examine the performance of the Milstein scheme for solving the stochastic Lorenz system. Where

- The drift is the usual deterministic part of the Lorenz system of the differential system of equations (refsdels).
- The diffusion matrix is given as:

$$b : \mathbb{T} \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}, \tag{25}$$

$$b\Big(t, x(t), y(t), z(t)\Big) = \lambda\Big(x(t) + y(t) + z(t)\Big) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Where $\lambda << 1$ is a stabilization factor.

Milstein Scheme for the stochastic Lorenz system

For constructing the numerical Milstein scheme, let us consider the three dimensional index set $F = 0, 1, 2, 3$, then the corresponding Hierarchical set $\mathcal{A}$ is

$$\mathcal{A} = \Big\{v, (0), (1), (2), (3), (1,1), (1,2), (1,3), (2,2), (2,1), (2,3), (3,3), (3,1), (3,2)\Big\}$$

The Milstein scheme can be derived from the stochastic Taylor expansion:

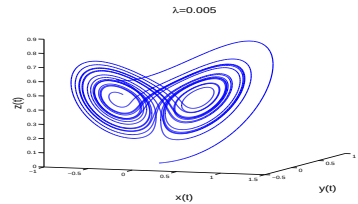
$$Y^1_{n+1} = Y^1_n + a^1(t_n, \mathbf{Y}_n)\Delta + \sum_{j=1}^{3} b^{1,j}(t_n, \mathbf{Y}_n)\Delta W^j_{t_n} + \sum_{j_1,j_2=1}^{3} L^{j_1} b^{1,j_2}(t_n, Y_n) I_{(j_1,j_2)}$$

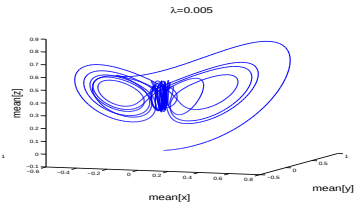
$$Y^2_{n+1} = Y^2_n + a^2(t_n, \mathbf{Y}_n)\Delta + \sum_{j=1}^{3} b^{2,j}(t_n, \mathbf{Y}_n)\Delta W^j_{t_n} + \sum_{j_1,j_2=1}^{3} L^{j_1} b^{2,j_2}(t_n, Y_n) I_{(j_1,j_2)}$$

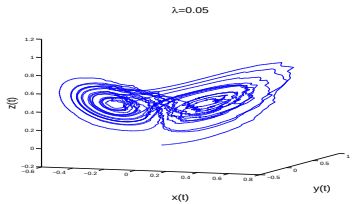
$$Y^3_{n+1} = Y^3_n + a^3(t_n, \mathbf{Y}_n)\Delta + \sum_{j=1}^{3} b^{1,j}(t_n, \mathbf{Y}_n)\Delta W^j_{t_n} + \sum_{j_1,j_2=1}^{3} L^{j_1} b^{3,j_2}(t_n, Y_n) I_{(j_1,j_2)}$$

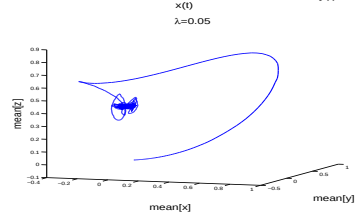
where Δ is a time step of the scheme. We deal here only with a fixed time step. Componentwise, we have

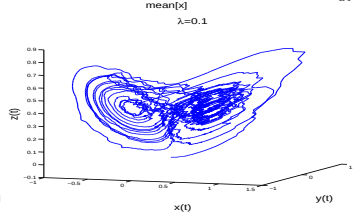
Simulation of the stochastic Lorenz system: The attractor

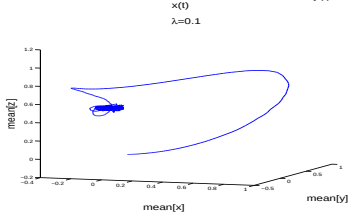






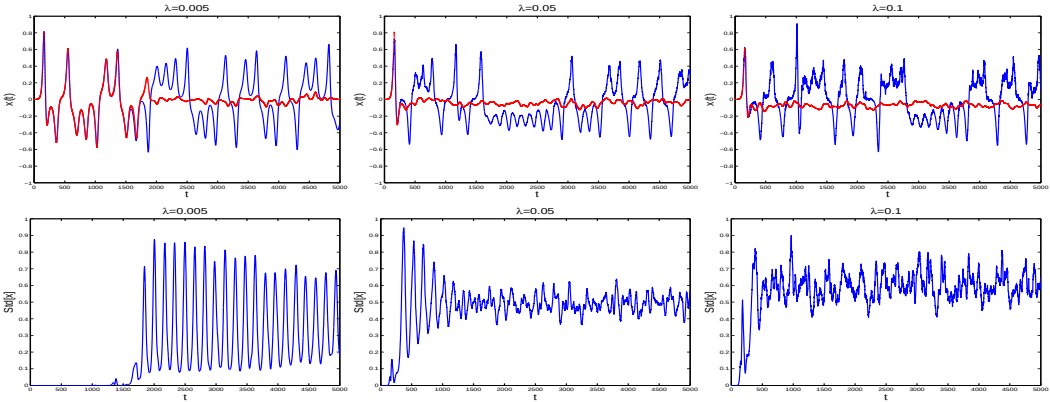






Simulation of the stochastic Lorenz system: Time series $x(t)$

It is important to note that the component of diffusion matrix (25) are correlated. Therefore, the fact that mean behavior of the SLA tends to converge to a stationary state.



(first row) First component ($x(t)$) of SLA (solid line) and the corresponding mean (red line). (second row) the corresponding standard deviation of $x(t)$.

Numerical Instability

- Although convergent in theory, a numerical scheme will be useful in practice unless it can **control the propagation of initially small errors**
- The Euler scheme for a constant time step

$$x_n = x_{n-1} - 10^N x_{n-1}\Delta = (1 - 10^N)x_{n-1} = \ldots = (1 - 10^N\Delta)^n x(0)$$

- For $\Delta > 2 \cdot 10^{-N}$, we get oscillation without bound: unstable scheme
- $x_n \to 0$ monotonically if $\Delta < 10^{-1}$. BUT for $N >> 1$ such time steps may be smaller than the round off resolution of the used computer$\varepsilon > 0$, there exist a $\delta \geqslant 0$such that

- Mean-square stable, if for each $\varepsilon > 0$, there exist a $\delta > 0$ such that

$$\|X(t; t_0, X_0)\|_{L^2}^2 < \varepsilon \quad t \geqslant t_0 \quad \text{whenever} \quad \|X_0\|_{L^2}^2 < \delta.$$

- Asymptotically mean-square stable, if it is mean-square stable and, when $\|X_0\|_{L^2}^2 < \delta$

$$\|X(t; t_0, X_0)\|_{L^2}^2 \longrightarrow 0$$

where $E(X_t) = \|X_t\|_{L^2}^2$ denotes the mean-square norm.

A nice stability analysis for the Milstein scheme is done by *V. Reshniak, A.Q.M. Khaliq, D.A.Voss, G.Zhang* **Split-step Milstein methods for multi-channel stiff stochastic differential systems**, Applied Numerical Mathematics 89 (2015) 1-23.

Concluding Remarks

- The Milstein scheme converges with strong order $\gamma = 1$ and weak order $\beta = 1$. It has a higher order of convergence in the strong sense than the Euler-Maruyama scheme, but gives no improvement in the weak sense.
- The Milstein scheme does **not correspond to a stochastic Taylor scheme** for an hierarchical set for the week convergence.
- **Pathwise convergence** was already considered for RODEs and is also interesting for SODEs because numerical calculations of the random variables Y_n in the numerical scheme above are carried out **path by path**. (P.E. Kloeden and A. Neuenkirch, The pathwise convergence of approximation schemes for stochastic differential equations, LMS J. Comp. Math. 10 (2007))
- Itô stochastic calculus is, however, an L^2 or a **mean-square calculus** and not a pathwise calculus.
- For the week convergence and their construction especially Hierarchical sets, we suggest the monograph of *Kloeden, Peter E., Platen, Eckhard* **Numerical Solution of Stochastic Differential Equations**.

Outlook Graduate-Lectures

- More details about stability, consistency and scheme properties
- Implicit schemes and relationship to stability
- Multistep stochastic schemes for SDEs
- Extrapolation (Richardson-like) stochastic schemes for SDEs
- Variance Reduction: Reduction of the number of sample for getting same results
- ...

References:


FWiersema U., Brownian Motion Calculus, John Wiley & Sons Ltd, 2008


Kloeden P. E., Platen E. Numerical Solution of Stochastic Differential Equations, Springer Verlag 1992.


Kloeden P. E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments, Springer Verlag 1993.


Milstein G. N. Numerical Integration of Stochastic Differential Equations, Kluwer Akademic Publishers 1995.


Karatzas I. and Shreve S. E. Brownian motion and stochastic calculus, Springer 1988.


Øksendal B. Stochastic Differential Equations, An Introduction with Applications, Springer Verlag 1985.


Desmond J. Higham, An Algorithmic Introduction to Numerical Simulation of StochasticDifferential Equations, SIAM Review, Vol. 43, No. 3, (2001),525–546.


K. Itô, Stochastic Integrals, Proc. Imp. Acad. Tokyo 20, 519–524 (1944)


K. Itô, Multiple Wiener Integral, J. Math. Soc. Japan 3, 157–169 (1951)


M. Zahri, Numerical Solution of a Stochastic Lorenz Attractor, Journal of Numerical Mathematics and Stochastics, 2 (1) : 01-11, 2010