

Energy-based model of self-organizing groups

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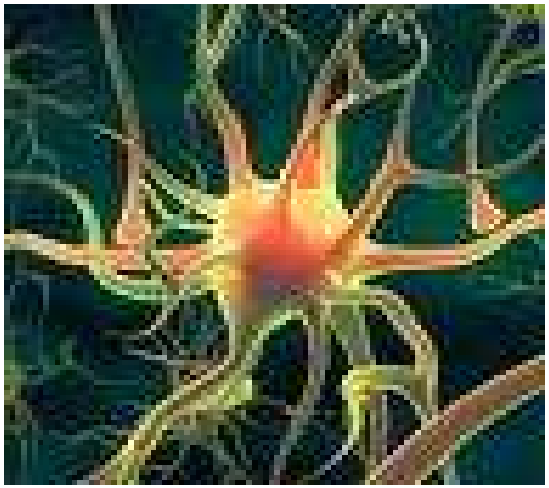
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Overview

- Introduction
- Finite metric space (FMS)
  - Energy-based ε−moves in a FMS
  - Convergence theorem
  - Simulation
- Continuous metric space (CMS)
  - Energy-based ε−moves in a CMS
  - Convergence theorem
  - Simulation
  - Simulation of segregation

INTRODUCTION: Condensing Phenomena, real life

Cells



Bees



Birds



- Self-Organization,
- Emergence,
- Non-linear systems,
- Distributive computing on networks
- Collective intelligence,
- **From individual to collective intelligence ?**

INTRODUCTION: Related and present works

1. Vicsek and al., Flocking, modeling and simulation (1995).
2. Hegselmann and Krause, Consensus, Opinion dynamics (2002).
3. Smale and Cucker, Flocking of birds, Math. of emergence (2006).
4. Birnir, Schooling of fishes (2007).

Present work:

1. Energy based model as extension of the HK-model.
2. From local interactions to global effect.
3. Analysis of the energies in a group.

FMS: Local and global energy map

Definition

(X, d) finite metric space, $m \in M_+(X)$ (nonnegative measures), $\varepsilon > 0$. The ε-energy of m is

$$E_\varepsilon(m) = \sum_{d(x,y)\leqslant\varepsilon} m(x)m(y)d^2(x,y).$$

The $\varepsilon-$ energy of point $a \in X$ with respect to m is

$$e_\varepsilon(a,m) = \sum_{d(a,y)\leqslant\varepsilon} m(y)d^2(a,y).$$

Note: $e_\varepsilon(\cdot, m)$ and $E_\varepsilon(m)$ are resp. local and global energies.

FMS: Moves of Point Mass

A pair $(a, a^*) \in X \times X$ operates on the set $M_+(X)$ as

$$m \to (a, a^*, m),$$

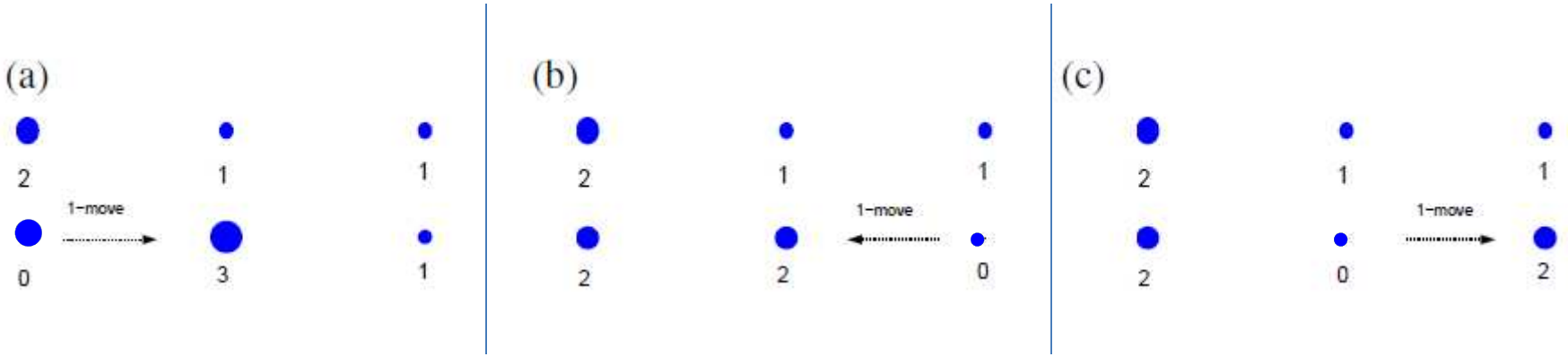
where

$$m^* = (a, a^*, m)(x) = \begin{cases} m(x); & \text{if} \quad x \notin \{a, a^*\}, \\ 0; & \text{if} \quad x = a, \\ m(a) + m(a^*); & \text{if} \quad x = a^*. \end{cases}$$

Note that

- when a moves to a^*, m moves to $m^* = (a, a^*, m)$, and
- $m(X) = m^*(X)$.

FMS: Example of moves



(a) m *is transformed to* m_a^* *with* $E(m) = E(m_a^*) = 6.$

(b) m *is transformed to* m_b^* *with* $E(m) = 6 < E(m_b^*) = 8.$

(c) m *is transformed to* m_c^* *with* $E(m) = 6 > E(m_c^*) = 0.$

We are looking for a local rule for a move of particles satisfying the case (c).

FMS: $\mathcal{E}$-Move: Definition

A pair (m, m^*) is $\varepsilon-$move, if there is a pair (a, a^*) such that:

$$
\begin{aligned}
&(i) \quad m^* = (a, a^*, m),\\
&(ii) \quad d(a, a^*) \leqslant \varepsilon,\\
&(iii) \quad e_\varepsilon(a, m) > e_\varepsilon(a^*, m^*).
\end{aligned}
$$

$m^1, m^2, \ldots$ is $\varepsilon-$condensing if (m^i, m^{i+1}) is $\varepsilon-$move.

Interpretation

Each agent *a* of the collection *m* is permanently looking for positions minimizing its local energy

FMS: Convergence theorem

Theorem

A singularly (one move at a time) $\varepsilon-$condensing sequence is finite. (in other words: The sequence converges.)

The proof is a consequence of the following lemmas.

Lemma

Let $m \in M_+(X), a, a^* \in X$ such that $d(a, a^*) \leqslant \varepsilon$. Then

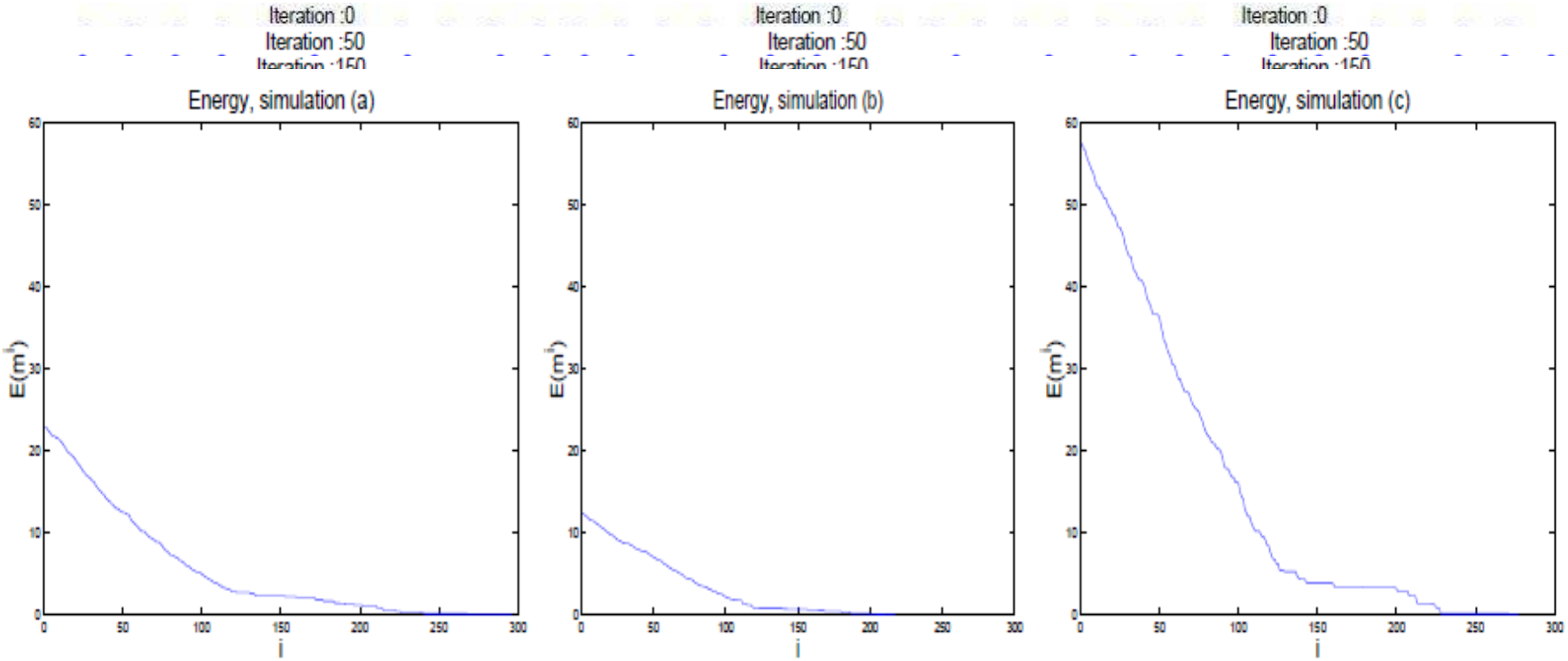
$$E_\varepsilon(m) - E_\varepsilon(m^*) = 2m(a)\left[e_\varepsilon(a, m) - e_\varepsilon(a^*, m) + m(a)d^2(a, a^*)\right],$$

Lemma

For $m \in M_+(X)$ let $n(m)$ be the number of elements $a \in X$ such that $m(a) > 0$. If $m^1, m^2, \ldots$ is a sequence of measures on X which is singularly and $\varepsilon-$condensing, then

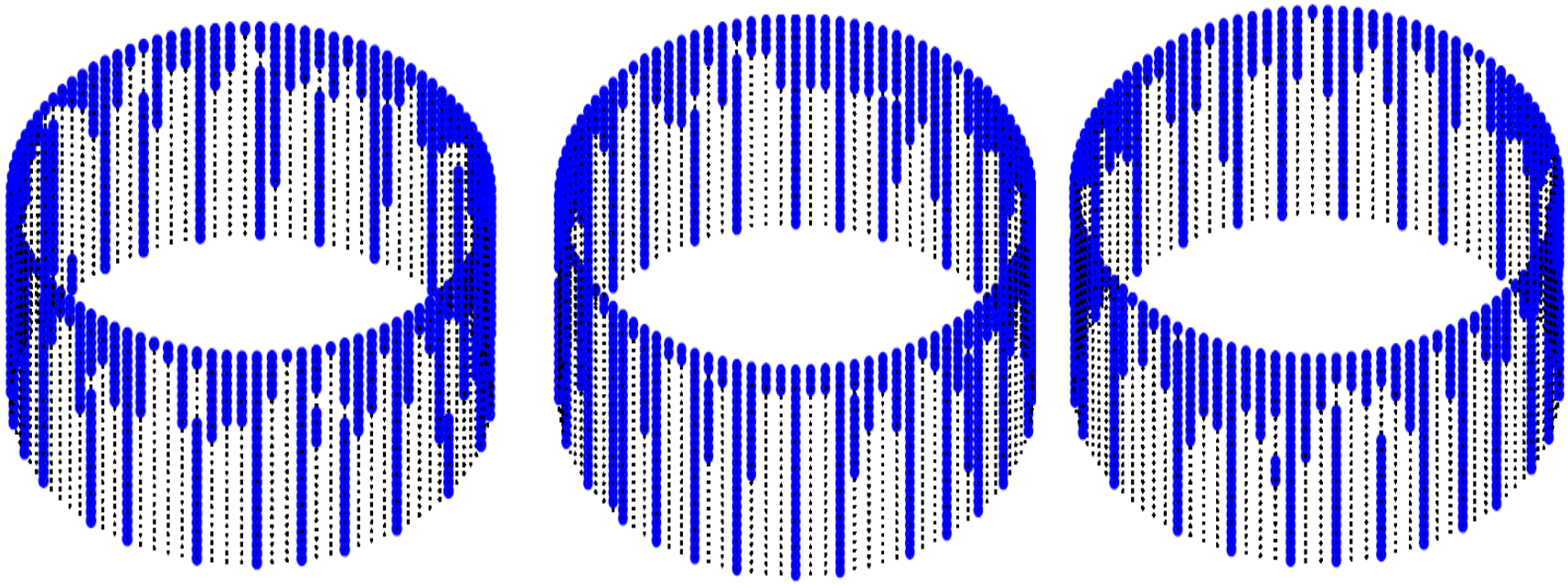
1. *$i \longrightarrow E_\varepsilon(m^i)$ is strictly decreasing*
2. *$i \longrightarrow n(m^i)$ is decreasing*

FMS: Simulation, Euclidean



Results of simulations in an (Euclidean) finite metric space.

FMS: Simulation, Circular



Results of simulations in a finite circular metric space.

CMS: Energy function

In general E is a not continuous function of m.

Example

For $X = \mathbb{R}$, $S(m) = \{1, 2\}$, $\varepsilon = 1$ and a given m with

$$m = \sum_{x \in \{1,2\}} m(x)\delta_x = \delta_1 + \delta_2 \Rightarrow E(m) = 2.$$

let m_j be a sequence of p.m. defined as $m_j = \delta_1 + \delta_{2+\frac{1}{j}}$, it follows $E(\lim_j m_j) = E(m) = 2 \neq \lim_j E(m_j) = 0$. Hence, the map E is not continuous in m.

CMS: Intensity function, Example

Definition

A continuous function φ, which satisfies:

$$\begin{aligned} \varphi &: X \times X \longrightarrow [0,1];\ \varepsilon, \theta > 0; \\ \varphi(x,y) &:= \begin{cases} 1, & \text{if } d(x,y) \leqslant \varepsilon, \\ 0, & \text{if } d(x,y) \geqslant \varepsilon + \theta. \end{cases} \end{aligned}$$

is called **intensity function**.

Example:

$$\begin{aligned} \varphi &: \mathbb{R} \times \mathbb{R} \longrightarrow [0,1];\ \varepsilon, \theta > 0; \\ \varphi(x,y) &:= \begin{cases} 1, & \text{if } |x-y| \leqslant \varepsilon, \\ 0, & \text{if } |x-y| \geqslant \varepsilon + \theta. \\ \left(\frac{\varepsilon+\theta-|x-y|}{\theta}\right) & \text{otherwise.} \end{cases} \end{aligned}$$

CMS: Continuous energy function

Definition

The continuous energy function of m is

$$\begin{aligned} E &: M_+(X) \longrightarrow \mathbb{R}^+ \\ E(m) &= \int_X \int_X \varphi(x,y) d^2(x,y) m(dx) m(dy) \\ &= \sum_{x,y} m(x) m(y) \varphi(x,y) d^2(x,y) \end{aligned}$$

The energy of point $a \in X$ with respect to m is

$$e(a,m) = \sum_y m(y) \varphi(a,y) d^2(a,y).$$

where φ is an intensity function.

CMS: Condensing sequence

Definition

A pair (m, m^*) is called condensing, if there is $(a, a^*) \in X \times X$ such that

(*i*) $m^* = (a, a^*, m)$ and

(*ii*) a^* minimizing $y \to e(y, m^*)\varphi(a, y)$, with $d(a, y) \leqslant \varepsilon + \theta$.

A sequence of non negative measures $m^1, m^2, \ldots$ is called condensing, if for every i the pair (m^i, m^{i+1}) is condensing. A mapping f such that $m^* = f(m)$ is called condensing.

CMS: Effectively condensing

Definition

A singularly condensing sequence $m^1, m^2, \ldots$ is called **effectively condensing sequence** if there exists $c > 0$ such that

(*i*) $m^{i+1} = (a, a^*, m^i)$,

(*ii*) $E(m^i) - E(m^{i+1}) \geqslant c\alpha(m^i)$,

where $\alpha(m^i) = \max_y \{\varphi(y, a)d^2(y, a) | y \in S(m^i)\}$.

From

$$\begin{aligned} E(m) - E(m^*) &= 2m(a)\left[e(a, m) - e(a^*, m^*)\right] \text{and} \\ c &= 2\min_a \{m(a) | a \in S(m^i)\}, \end{aligned}$$

follow the existence of the effectively condensing sequence.

CMS: Convergence theorem

Theorem

If (X, d) is a compact metric space, then every effectively condensing sequence $m^1, m^2, \ldots$ is finite.

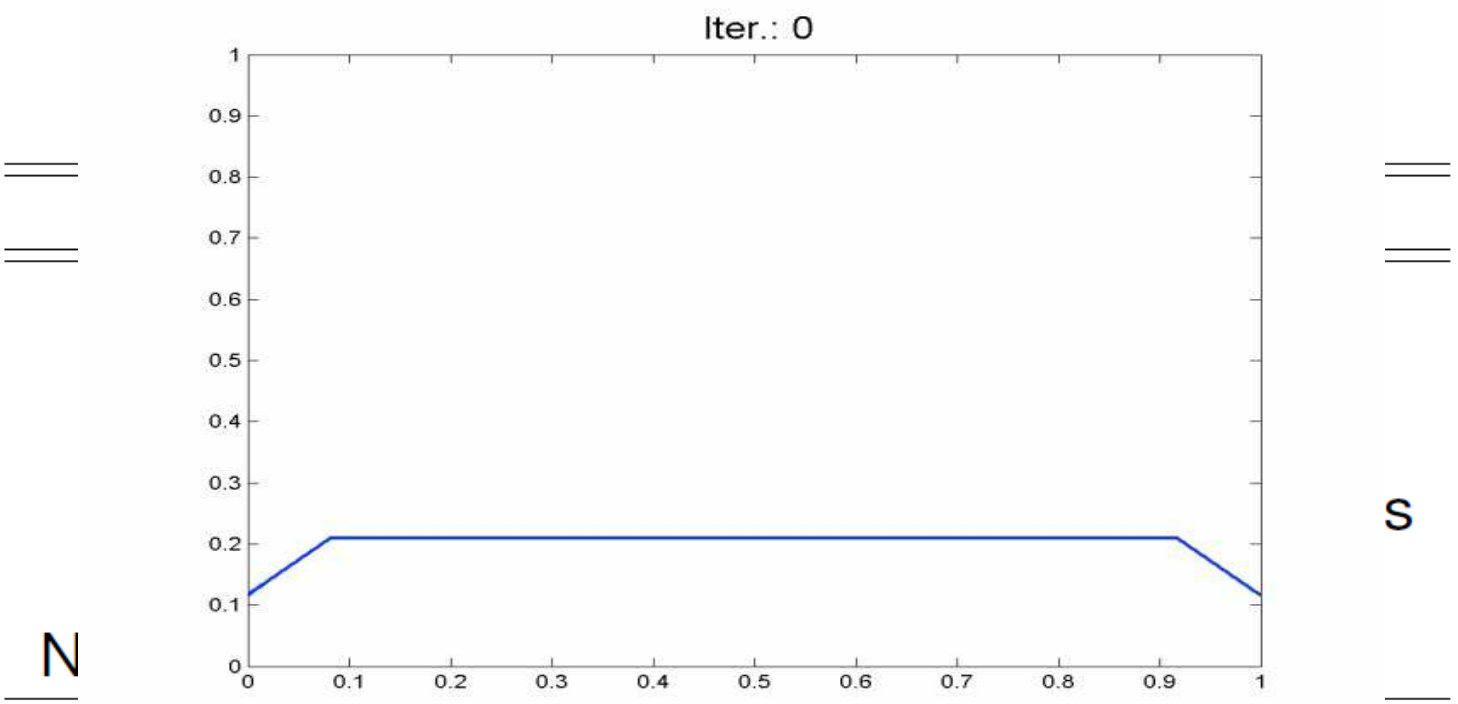
Proof idea: Since X is compact and

Lemma

Suppose that $m^1, m^2, \ldots$ is an effectively condensing.

$$\text{if } \lim_{i \to \infty} E(m^i) = E(m), \text{ then } \alpha(m) = 0.$$

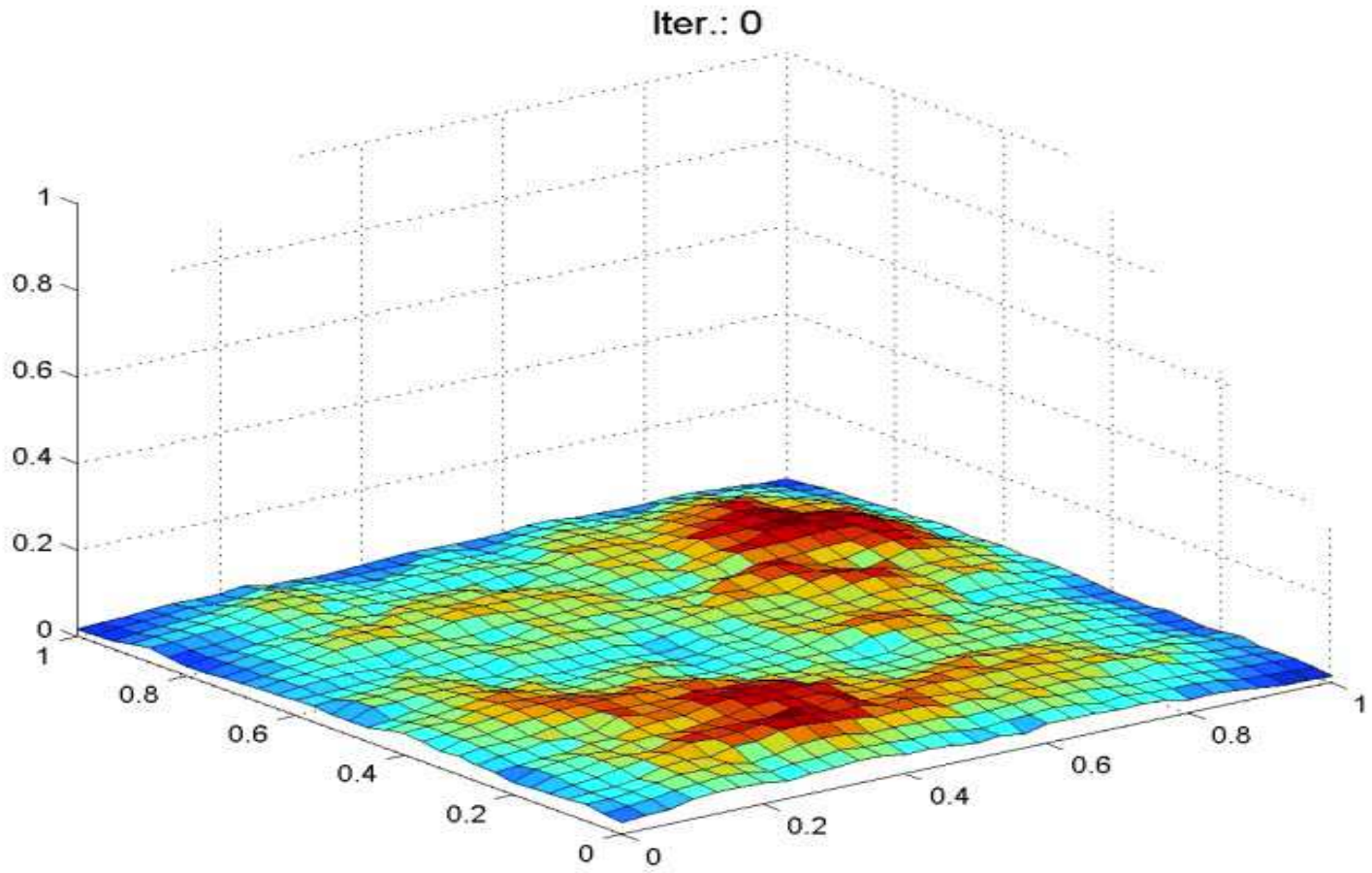
CMS: Simulation, real Line



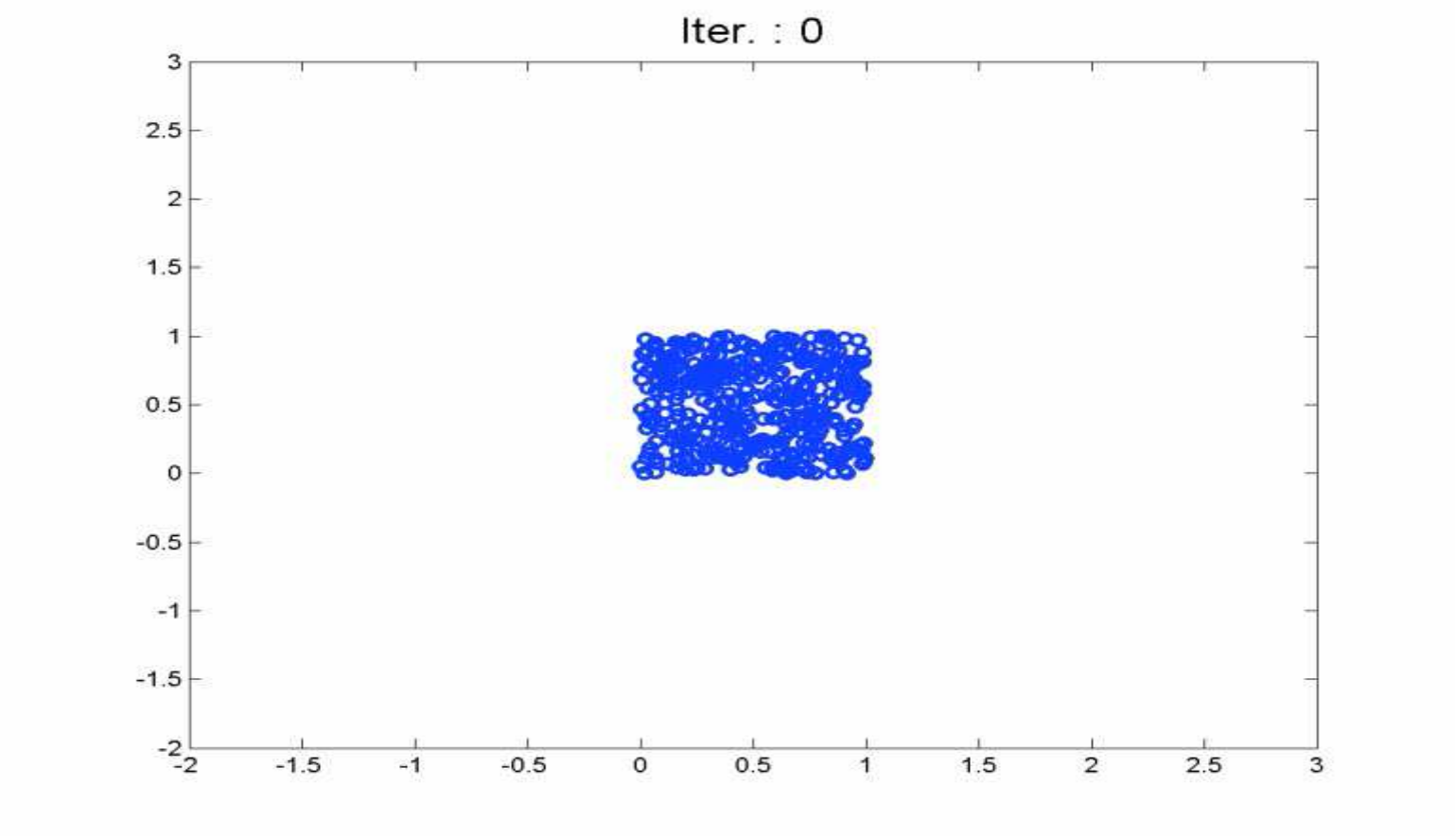
Results of simulations in the real line.

CMS: Simulation, real plane

Two dimensional density function of a condensing sequence of measures



CMS: Simulation, Segregation



Prohibition of collisions leads to segregation

Danke