

Condensing on metric spaces Modeling, Analysis and Simulation

Mostafa Zahri

Frankfurt 31.07.2009

Overview

- Introduction
- Finite metric space
  - Definition of $\varepsilon-$moves and condensing sequences
  - Convergence theorem
  - Simulation of condensing sequences
- Continuous metric space
  - Definition of $\varepsilon-$moves and condensing sequences
  - Convergence theorem
  - Simulation
  - Simulation of segregation

Introduction

Condensing Phenomena

Cells



Bees



Birds



- Self-Organization,
- Emergence,
- Non-linear systems,
- Distributive computing on networks
- Collective intelligence,
- **From individual to collective intelligence ?**

Students



Introduction

Hegselmann-Krause-model (HK-model)

The HK-model is a process of consensus formation with bounded confidence: A systems of points for $\varepsilon > 0$ and $t = 0, 1, 2, \ldots$

$$\begin{aligned} x_i(t+1) &= \frac{1}{\sharp I(i,t)} \sum_{j \in I(i,t)} x_j(t) \\ I(i,t) &= \{j \,\big|\big|\, |x_i(t) - x_j(t)| \leqslant \varepsilon\} \end{aligned}$$

Consensus is the state (c is constant)

$$\lim_{t \to \infty} x_i(t) = c.$$

The HK-model is also known as OPINION DYNAMICS.

Introduction

Energy-based-model

Starting from the ε-energy of a position resp. to m (pos. measure):

$$e_\varepsilon(x, m) = \int_{d(x,y)\leqslant\varepsilon} d^2(x, y)m(dy),$$

-extending the HK-model-, we

1. replace the Euclidean space by a metric space (X, d),
2. let the agents move to where the local energy is minimal,
3. Agents do not move simultaneously,

and observe the total ε-energy of m

$$E_\varepsilon(m) = \int_X e_\varepsilon(x, m)m(dx),$$

Finite metric space

Local and global energy map

Definition

(X, d) finite metric space, $m \in M_+(X)$ (nonnegative measures), $\varepsilon > 0$. The ε-energy of m is

$$E_\varepsilon(m) = \sum_{d(x,y)\leqslant\varepsilon} m(x)m(y)d^2(x,y).$$

The $\varepsilon-$ energy of point $a \in X$ with respect to m is

$$e_\varepsilon(a,m) = \sum_{d(a,y)\leqslant\varepsilon} m(y)d^2(a,y).$$

Note: $e_\varepsilon(\cdot, m)$ and $E_\varepsilon(m)$ are resp. local and global energies.

Finite metric space
Moves of Point Mass

A pair $(a, a^*) \in X \times X$ operates on the set $M_+(X)$ as

$$m \to (a, a^*, m),$$

where

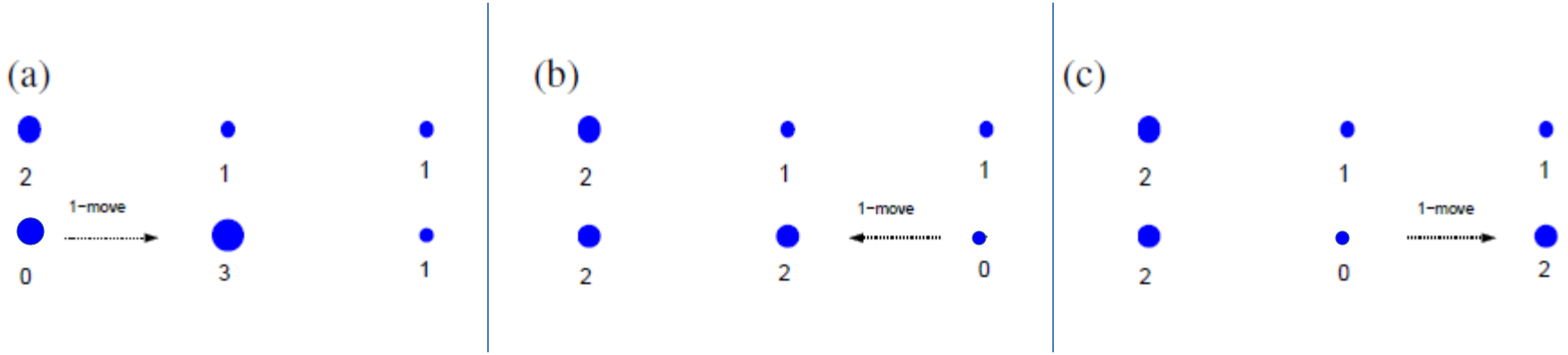
$$m^* = (a, a^*, m)(x) = \begin{cases} m(x); & \text{if} \quad x \notin \{a, a^*\}, \\ 0; & \text{if} \quad x = a, \\ m(a) + m(a^*); & \text{if} \quad x = a^*. \end{cases}$$

Note that

- when a moves to a^*, m moves to $m^* = (a, a^*, m)$, and
- $m(X) = m^*(X)$.

Finite metric space

Example of moves



(a) m *is transformed to* m_a^* *with* $E(m) = E(m_a^*) = 6.$

(b) m *is transformed to* m_b^* *with* $E(m) = 6 < E(m_b^*) = 8.$

(c) m *is transformed to* m_c^* *with* $E(m) = 6 > E(m_c^*) = 0.$

We are looking for a local rule for a move of particles satisfying the case (c).

ε-MOVE: DEFINITION

A pair (m, m^*) is $\varepsilon-$move, if there is a pair (a, a^*) such that:

$$
\begin{aligned}
&(i) \quad m^* = (a, a^*, m),\\
&(ii) \quad d(a, a^*) \leqslant \varepsilon,\\
&(iii) \quad e_\varepsilon(a, m) + m(a)d^2(a, a^*) > e_\varepsilon(a^*, m).
\end{aligned}
$$

$m^1, m^2, \ldots$ is $\varepsilon-$condensing if (m^i, m^{i+1}) is $\varepsilon-$move.

INTERPRETATION

a moves to a^* if $d(a^*, a) \leqslant \varepsilon$ and $e(a^*, m^*) < e(a, m)$.

Finite metric space
Convergence theorem

Theorem

A singularly $\varepsilon-$condensing sequence is finite.

The proof is a consequence of the following lemmas.

Lemma

Let $m \in M_+(X), a, a^* \in X$ such that $d(a, a^*) \leqslant \varepsilon$. Then

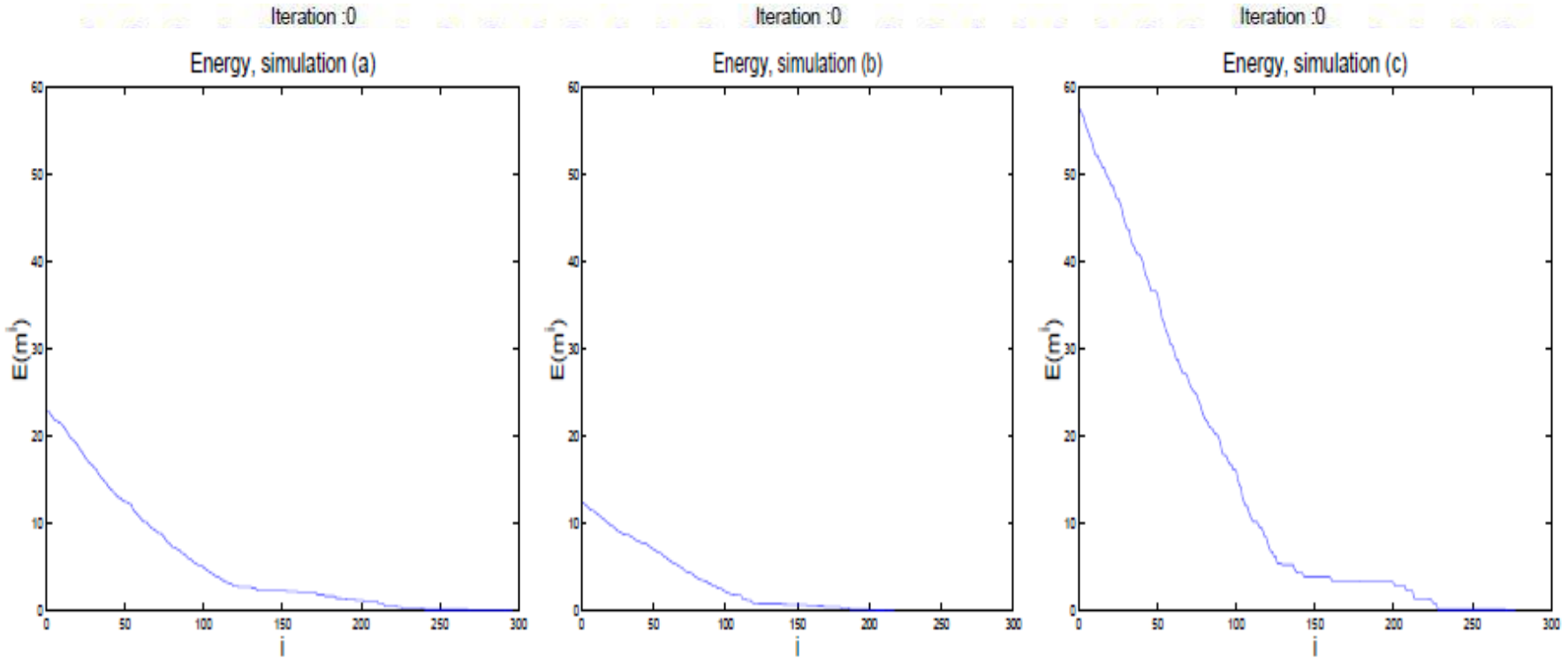
$$E_\varepsilon(m) - E_\varepsilon(m^*) = 2m(a)\left[e_\varepsilon(a, m) - e_\varepsilon(a^*, m) + m(a)d^2(a, a^*)\right],$$

Lemma

For $m \in M_+(X)$ let $n(m)$ be the number of elements $a \in X$ such that $m(a) > 0$. If $m^1, m^2, \ldots$ is a sequence of measures on X which is singularly and $\varepsilon-$condensing, then

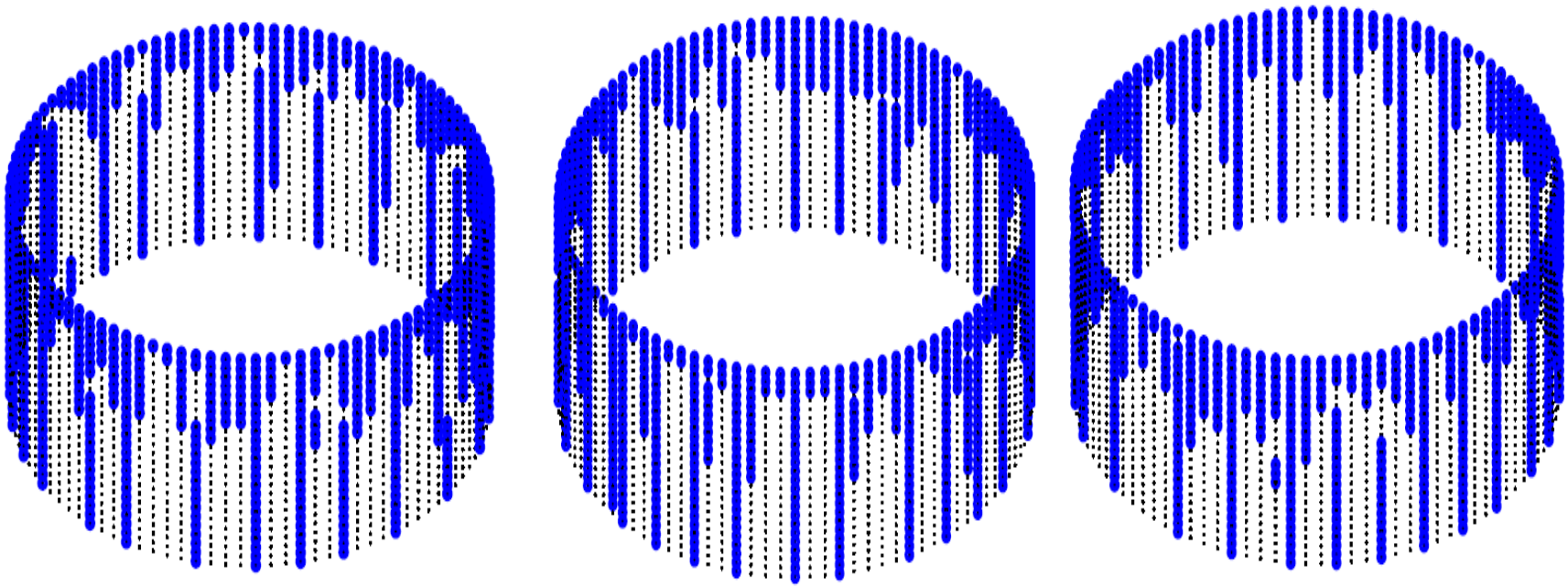
1. *$i \longrightarrow E_\varepsilon(m^i)$ is strictly decreasing*
2. *$i \longrightarrow n(m^i)$ is decreasing*

Finite metric space
Simulation, Euclidean



Results of simulations in an (Euclidean) finite metric space.

Finite metric space
Simulation, Circular



Results of simulations in a finite circular metric space.

Continuous metric space
Energy function

In general E is a not continuous function of m.

Example

For $X = \mathbb{R}$, $S(m) = \{1, 2\}$, $\varepsilon = 1$ and a given m with

$$m = \sum_{x \in \{1,2\}} m(x)\delta_x = \delta_1 + \delta_2 \Rightarrow E(m) = 2.$$

let m_j be a sequence of p.m. defined as $m_j = \delta_1 + \delta_{2+\frac{1}{j}}$, it follows $E(\lim_j m_j) = E(m) = 2 \neq \lim_j E(m_j) = 0$. Hence, the map E is not continuous in m.

Continuous metric space

Intensity function, Example

Definition

A continuous function φ, which satisfies:

$$\begin{aligned} \varphi &: X \times X \longrightarrow [0,1];\ \varepsilon, \theta > 0; \\ \varphi(x,y) &:= \begin{cases} 1, & \text{if } d(x,y) \leqslant \varepsilon, \\ 0, & \text{if } d(x,y) \geqslant \varepsilon + \theta. \end{cases} \end{aligned}$$

is called **intensity function**.

Example:

$$\begin{aligned} \varphi &: \mathbb{R} \times \mathbb{R} \longrightarrow [0,1];\ \varepsilon, \theta > 0; \\ \varphi(x,y) &:= \begin{cases} 1, & \text{if } |x-y| \leqslant \varepsilon, \\ 0, & \text{if } |x-y| \geqslant \varepsilon + \theta. \\ \left(\frac{\varepsilon+\theta-|x-y|}{\theta}\right) & \text{otherwise.} \end{cases} \end{aligned}$$

Continuous metric space

Continuous energy function

Definition

The continuous energy function of m is

$$\begin{aligned} E &: M_+(X) \longrightarrow \mathbb{R}^+ \\ E(m) &= \int_X \int_X \varphi(x,y) d^2(x,y) m(dx) m(dy) \\ &= \sum_{x,y} m(x) m(y) \varphi(x,y) d^2(x,y) \end{aligned}$$

The energy of point $a \in X$ with respect to m is

$$e(a,m) = \sum_y m(y) \varphi(a,y) d^2(a,y).$$

where φ is an intensity function.

Continuous metric space

Condensing sequence

Definition

A pair (m, m^*) is called condensing, if there is $(a, a^*) \in X \times X$ such that

(*i*) $m^* = (a, a^*, m)$ and

(*ii*) a^* minimizing $y \to e(y, m^*)\varphi(a, y)$, with $d(a, y) \leqslant \varepsilon + \theta$.

A sequence of non negative measures $m^1, m^2, \ldots$ is called condensing, if for every i the pair (m^i, m^{i+1}) is condensing. A mapping f such that $m^* = f(m)$ is called condensing.

Continuous metric space
Effectively condensing

Definition

A singularly condensing sequence $m^1, m^2, \ldots$ is called **effectively condensing sequence** if there exists $c > 0$ such that

(*i*) $m^{i+1} = (a, a^*, m^i)$,

(*ii*) $E(m^i) - E(m^{i+1}) \geqslant c\alpha(m^i)$,

where $\alpha(m^i) = \max_y \{\varphi(y, a)d^2(y, a) | y \in S(m^i)\}$.

From

$$\begin{aligned} E(m) - E(m^*) &= 2m(a)\left[e(a, m) - e(a^*, m^*)\right] \text{and} \\ c &= 2\min_a \{m(a) | a \in S(m^i)\}, \end{aligned}$$

follow the existence of the effectively condensing sequence.

Continuous metric space
Convergence of effectively Cond. Seq.

Theorem

If (X, d) is a compact metric space, then every effectively condensing sequence $m^1, m^2, \ldots$ is finite.

Proof idea: Since X is compact and

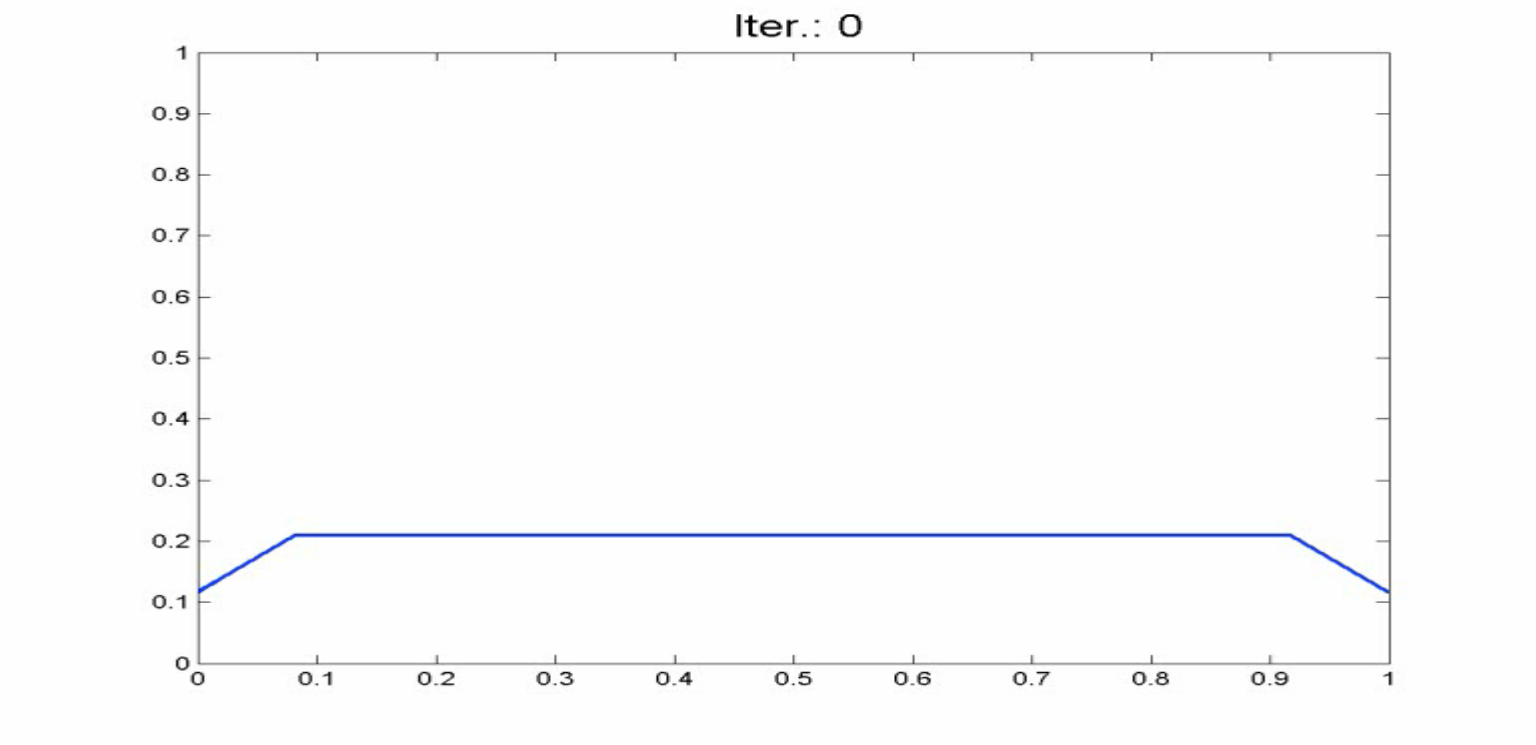
Lemma

Suppose that $m^1, m^2, \ldots$ is an effectively condensing.

$$\text{if } \lim_{i \to \infty} E(m^i) = E(m), \text{ then } \alpha(m) = 0.$$

Continuous metric space

Simulation, real Line

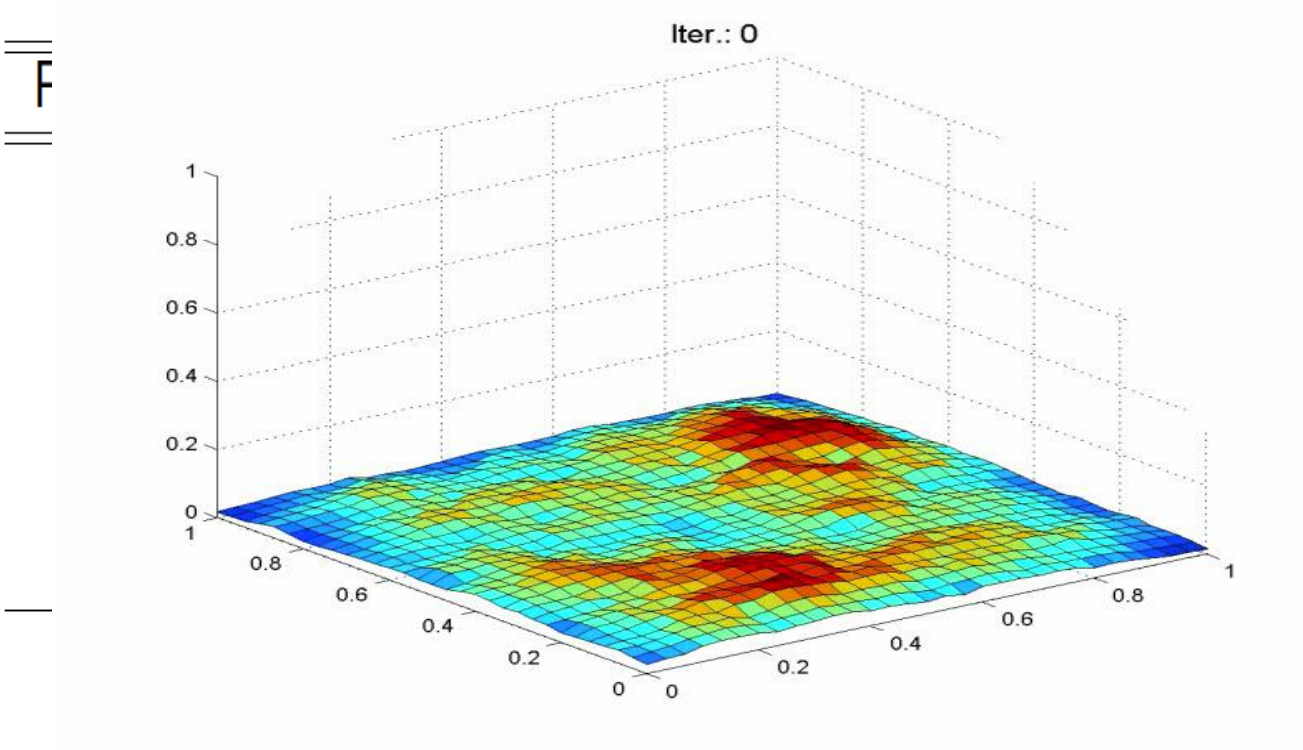


Results of simulations in the real line.

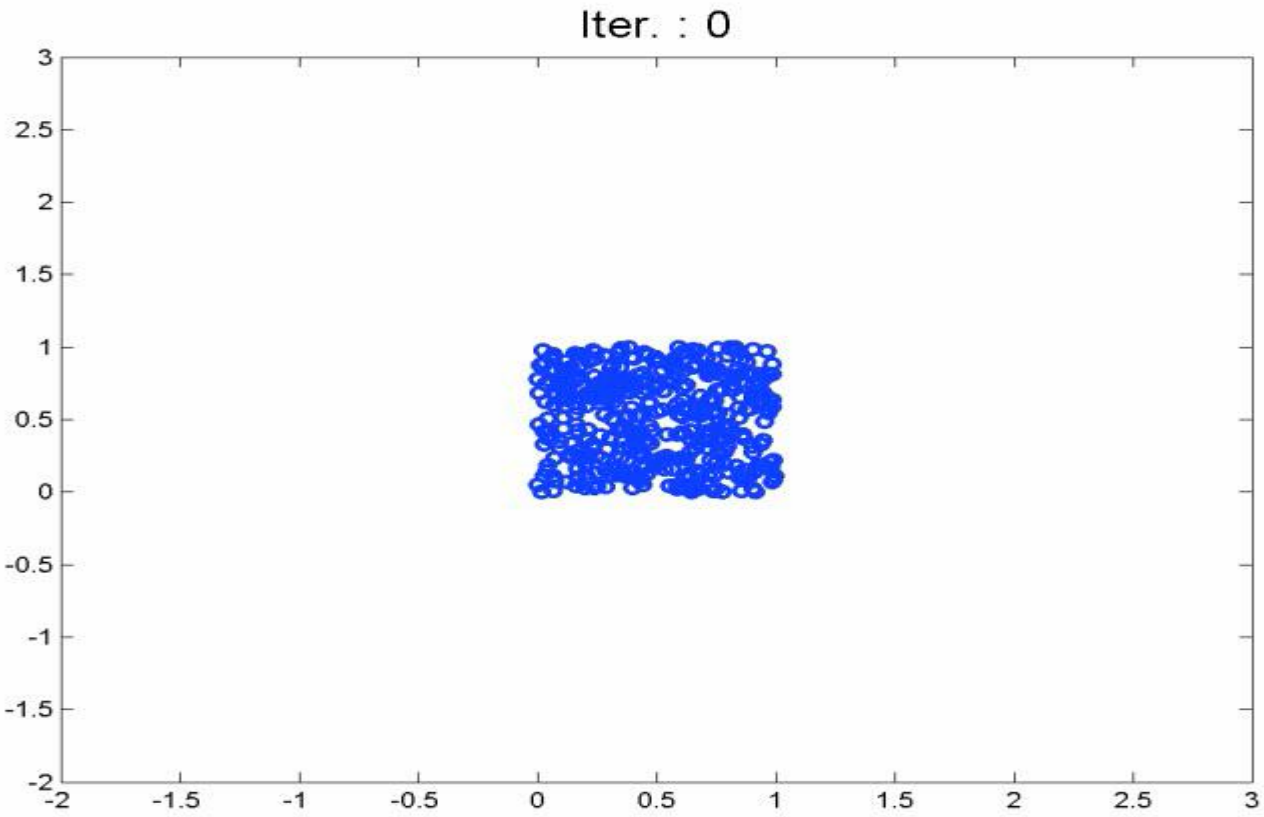
Continuous metric space

Simulation, real plane

Two dimensional density function of a condensing sequence of measures



Continuous metric space
Simulation, Segregation



Prohibition of collisions leads to segregation

Danke